Some remarks on Banach-Colmez spaces and related diamonds

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1 Setup

Let C be an algebraically closed perfectoid field over $\mathbf{F}_p$, and let $\mathcal{X}_C$ denote the adic Fargues-Fontaine curve over C. One of the 2017 AWS projects is to classify short exact sequences

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of vector bundles over $\mathcal{X}_C$. In particular, for a given pair $(\mathcal{E}_1, \mathcal{E}_2)$, we'd like to enumerate the set of isomorphism classes of $\mathcal{E}$'s which can appear as such an extension. There are some evident necessary conditions: for example, there is an inequality $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ (cf. Kiran's notes for a proof). To what degree are the necessary conditions sufficient?

1.1 A zoo of diamonds

Here we sketch a strategy for dealing with a supplementary issue, whose answer should play an important role in any treatment of this problem: Given two bundles $\mathcal{E}$ and $\mathcal{F}$, determine when $\mathcal{F}$ is a quotient of $\mathcal{E}$, and determine the dimension of the space of surjective maps $\mathcal{E} \twoheadrightarrow \mathcal{F}$. It turns out that a nice dichotomy occurs: either there are no surjective maps $\mathcal{E} \to \mathcal{F}$, or the space of such maps is equidimensional of an easily calculated dimension. The nontrivial matter is to determine which case one is in.

Two words on terminology: 1) For the definition of a diamond, see Jared's lecture notes. For the notions of locally spatial diamonds and smooth (morphisms of) diamonds, one can e.g. look at §4.3 of this or at these notes. 2) In what follows, we define the dimension $\dim X$ of a locally spatial diamond X as the Krull dimension of the locally spectral topological space $|X|$. We say X is equidimensional of dimension d if $\dim U = d$ for any open subdiamond $U \subset X$.

First let's define some functors on $\mathrm{Perf}_{/\mathrm{Spa}\, C}$. First, given a vector bundle $\mathcal{E}$ on $\mathcal{X}_C$, let $\mathcal{H}^0(\mathcal{E}) : \mathrm{Perf}_{/\mathrm{Spa}\, C} \to \mathrm{Sets}$ be the functor sending $f : S \to \mathrm{Spa}\, C$ to the set $H^0(\mathcal{X}_S, \mathcal{E}_S)$, where $\mathcal{E}_S$ is the pullback of $\mathcal{E}$ along the canonical map $\mathcal{X}_S \to \mathcal{X}_C$ induced by f. Likewise, given two vector bundles $\mathcal{E}$ and $\mathcal{F}$, define $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ as the functor sending $f : S \to \mathrm{Spa}\, C$ to the set of $\mathcal{O}_{\mathcal{X}_S}$-module maps $m : \mathcal{E}_S \to \mathcal{F}_S$. Note that $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee \otimes \mathcal{F})$.

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Next, let $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F}) \subset \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ be the subfunctor parametrizing surjective $\mathcal{O}_{\mathcal{X}_S}$-module maps. Dually, let $\mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F}) \subset \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ be the subfunctor of "fiberwise-injective" $\mathcal{O}_{\mathcal{X}_S}$-module maps: precisely, this is the functor parametrizing $\mathcal{O}_{\mathcal{X}_S}$-module maps $m : \mathcal{E}_S \to \mathcal{F}_S$ such that for every geometric point $x : \mathrm{Spa}(C', C'^+) \to S$, the pullback of $m : \mathcal{E}_S \to \mathcal{F}_S$ along $\mathcal{X}_{C',C'^+} \to \mathcal{X}_S$ gives an injective $\mathcal{O}_{\mathcal{X}_{C,C'^+}}$-module map.[1] Finally, let $\mathcal{H}^0(\mathcal{E})^\times := \mathcal{I}\mathrm{nj}(\mathcal{O},\mathcal{E}) \subset \mathcal{H}^0(\mathcal{E})$ be the functor parametrizing sections of $\mathcal{E}$ which are not identically zero on any fiber of $\mathcal{X}_S \dashrightarrow S$.

The first order of business is to check that all of these functors are nice locally spatial diamonds.

Lemma 1.1. *$\mathcal{H}^0(\mathcal{E})$ is a partially proper locally spatial "$\mathbf{Q}_p$-vector diamond" over $\mathrm{Spd}\, C$, equidimensional of dimension $\deg(\mathcal{E}^{\geq 0})$. Each connected component of $\mathcal{H}^0(\mathcal{E})$ is partially proper and smooth.*

Here $\mathcal{E}^{\geq 0} \subset \mathcal{E}$ denotes the evident step in the HN-filtration of $\mathcal{E}$.[2] Note that "partial properness" here is a formal consequence of the fact that the category of vector bundles on $\mathcal{X}_{\mathrm{Spa}(R,R^+)}$ is canonically independent of R^+; in fact, every functor defined above (and below) is partially proper for this reason.

Proof. Use the classification together with the isomorphism $\mathcal{H}^0(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \mathcal{H}^0(\mathcal{E}_1) \times_{\mathrm{Spd}\, C} \mathcal{H}^0(\mathcal{E}_2)$ to reduce to the case $\mathcal{E} = \mathcal{O}(\lambda)$ for $\lambda \in \mathbf{Q}_{\geq 0}$, and then argue by hand. □

Lemma 1.2. *For any $\mathcal{E},\mathcal{F}$, $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})$ is an open subfunctor of $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee \otimes \mathcal{F})$. In particular, $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})$ is a (partially proper) locally spatial diamond over $\mathrm{Spd}\, C$, which is either empty or equidimensional of dimension*

$$\dim \mathcal{H}^0(\mathcal{E}^\vee \otimes \mathcal{F}) = \deg{(\mathcal{E}^\vee \otimes \mathcal{F})}^{\geq 0}.$$

Proof. Choose some $T \in \mathrm{Perf}$ together with a surjective pro-etale morphism $T^\diamond \to \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$. Over $\mathcal{X}_T$, we get a "universal" $\mathcal{O}_{\mathcal{X}_T}$-module map $m^{\mathrm{univ}} : \mathcal{E}_T \to \mathcal{F}_T$; let $\mathcal{Q}$ be the cokernel of m^{univ}. By a standard argument, the support of $\mathcal{Q}_T$ is Zariski-closed in $\mathcal{X}_T$, and we write $Z \subset |\mathcal{X}_T|$ for the associated closed subset.

Next, we observe that the map $|\mathcal{X}_T| \to |T|$ is closed. Indeed, this is a specializing quasicompact spectral map of locally spectral spaces, so the image of any closed subset is pro-constructible (by quasicompacity and spectrality) and stable under specialization, hence closed by [Stacks, Tag 0903]. In particular, the subset $V = \mathrm{im}(|\mathcal{X}_T| \to |T|)(Z) \subset |T|$ is closed. We also observe that a geometric point $x : \mathrm{Spd}(C', C'^+) \to \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ defines a point of $\mathcal{S}\mathrm{urj}$ (resp. $\mathcal{H}\mathrm{om} \smallsetminus \mathcal{S}\mathrm{urj}$) if and only if the preimage of $|x|$ in $|T|$ is disjoint from V (resp. contained in V). In particular, the open subset $U = |T| \smallsetminus V \subset |T|$ is the preimage of a subset $W \subset |\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})|$; since $|T| \twoheadrightarrow |\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})|$ is a quotient map, W is open. But now $\mathcal{S}\mathrm{urj}$ can be identified with the open subdiamond of $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ corresponding to the open subset W, so we win. □

In particular, $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})$ is either empty or "big" inside $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$. The same results hold for $\mathcal{I}\mathrm{nj}$, though with a rather different proof.

[1] The condition defining $\mathcal{I}\mathrm{nj}$ is stronger than the mere condition that $m : \mathcal{E}_S \to \mathcal{F}_S$ be injective. Note that the association sending S to the set of injective m's isn't even a presheaf. Note also that there is a natural transformation $\mathcal{S}\mathrm{urj}(\mathcal{F}^\vee,\mathcal{E}^\vee) \to \mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F})$; this turns out to be an open immersion, although it typically isn't an isomorphism.

[2] More generally, if $\mathcal{E}$ has HN filtration $0 \subsetneq \mathcal{E}^1 \subsetneq \mathcal{E}^2 \subsetneq \cdots \subsetneq \mathcal{E}^n = \mathcal{E}$ with corresponding slopes $\mu_1 > \mu_2 > \cdots > \mu_n$, we define $\mathcal{E}^{\geq \lambda}$ (resp. $\mathcal{E}^{>\lambda}$) as $\mathcal{E}^i$ where i is chosen maximally among those indices with $\mu_i \geq \lambda$ (resp. with $\mu_i > \lambda$). We also set $\mathcal{E}_{<0} = \mathcal{E}/\mathcal{E}^{\geq 0}$ and $\mathcal{E}_{\leq 0} = \mathcal{E}/\mathcal{E}^{>0}$.

Lemma 1.3. *The functor* $\mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F})$ *is an open subfunctor of* $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$*. In particular,* $\mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F})$ *is a (partially proper) locally spatial diamond over* $\operatorname{Spd} C$*, empty or equidimensional of dimension* $\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0}$.

Proof. Set $r = \operatorname{rank}(\mathcal{E})$; by the formula

$$\mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F}) \cong \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F}) \times_{\mathcal{H}\mathrm{om}(\wedge^r\mathcal{E},\wedge^r\mathcal{F})} \mathcal{I}\mathrm{nj}(\wedge^r\mathcal{E},\wedge^r\mathcal{F}),$$

we reduce to the case where $\mathcal{E}$ is a line bundle. After twisting, we reduce further to the case $\mathcal{E} = \mathcal{O}$; in other words, we need to prove that $\mathcal{H}^0(\mathcal{F})^\times$ is an open subfunctor of $\mathcal{H}^0(\mathcal{F})$. Fix an identification $\mathcal{F} = \oplus_{1\leq i\leq n}\mathcal{O}(\lambda_i)$, and (for brevity) set $\mathcal{H}_i = \mathcal{H}^0(\mathcal{O}(\lambda_i))$ and $\mathcal{H}_i^\times = \mathcal{H}^0(\mathcal{O}(\lambda_i))^\times$. Under the identification

$$\mathcal{H}^0(\mathcal{F}) = \mathcal{H}_1 \times_{\operatorname{Spd} C} \cdots \times_{\operatorname{Spd} C} \mathcal{H}_n,$$

it is easy to see that the subfunctor $\mathcal{H}^0(\mathcal{F})^\times$ on the left-hand side is covered by the union of the subfunctors

$$\mathcal{U}_i = \mathcal{H}_1 \times \cdots \times \mathcal{H}_{i-1} \times \mathcal{H}_i^\times \times \mathcal{H}_{i+1} \times \cdots \times \mathcal{H}_n,\ 1 \leq i \leq n$$

on the right-hand side (here we've omitted the subscripted $\operatorname{Spd} C$'s for brevity). This, finally, reduces us to showing that $\mathcal{H}^0(\mathcal{O}(\lambda_i))^\times$ is an open subfunctor of $\mathcal{H}^0(\mathcal{O}(\lambda_i))$, which can be checked by hand. □

Note that $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E},\mathcal{F})$ are *not* sub-Banach-Colmez spaces of $\mathcal{H}^0(\mathcal{E}^\vee\otimes\mathcal{F})$, because they don't have any natural vector space structure. This is basically why we need to bring the full power of diamonds into the picture if we'd like to make "dimension arguments" in the setting of objects like $\mathcal{S}\mathrm{urj}$ and $\mathcal{I}\mathrm{nj}$: the dimension theory for Banach-Colmez spaces developed by Colmez isn't directly applicable.

OK, here's the first real goal.

Theorem 1.4. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be given. Suppose that* $\operatorname{rank}(\mathcal{E}) > \operatorname{rank}(\mathcal{F})$ *and that* $\mathcal{F}$ *is semistable of slope* $\geq$ *the maximal slope of* $\mathcal{E}$*. Then* $\mathcal{F}$ *is a quotient of* $\mathcal{E}$*, i.e.* $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})(C)$ *is nonempty.*

From this, one easily deduces that the same result holds more generally when $\mathcal{F}$ is semistable of some slope λ and $\operatorname{rank}(\mathcal{E}_{\leq\lambda}) > \operatorname{rank}(\mathcal{F})$, and that the latter condition is necessary for such an $\mathcal{F}$ to arise as a quotient of $\mathcal{E}$ (except in the degenerate case where $\mathcal{E}_{\leq\lambda} \simeq \mathcal{F}$). In particular, this theorem is optimal.

Let me sketch a strategy for proving this theorem. To be clear, I haven't checked all the details of the following arguments yet, but the most interesting remaining steps should "just" be combinatorics. The idea is as follows. We are not going to exhibit any particular surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$. Instead, the strategy is to calculate $d_1 = \dim \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ and $d_2 = \dim(\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F}) \smallsetminus \mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F}))$ explicitly, and then simply observe that $d_2 < d_1$ when the hypotheses of the theorem hold. Note that $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F}) \smallsetminus \mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{F})$ is a nice closed subdiamond of $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$. In order to calculate d_2, we stratify $\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$ according to the isomorphism type of the bundle $\operatorname{im}(\mathcal{E} \stackrel{m_x}{\to} \mathcal{F})$ where $m_x : \mathcal{E} \to \mathcal{F}$ is the map parametrized by a (geometric) point $x \in \mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{F})$.

Precisely, let $Quo(\mathcal{E},\mathcal{F})$ denote the set of isomorphism classes of (nonzero) quotient bundles $\mathcal{Q}$ of $\mathcal{E}$ which also occur as subsheaves of $\mathcal{F}$. A priori, we don't know very much about this set; after all, we're trying to show that $\mathcal{F} \in Quo(\mathcal{E},\mathcal{F})$! However, slope theory puts strong constraints on the possible elements of $Quo(\mathcal{E},\mathcal{F})$. For example, any such $\mathcal{Q}$ has rank $\leq \operatorname{rank}\mathcal{F}$, minimal slope $\geq$ the minimal slope of $\mathcal{E}$, and maximal slope $\leq$ the maximal slope of $\mathcal{F}$; these constraints immediately imply the finiteness of $Quo(\mathcal{E},\mathcal{F})$.

For each (choice of a bundle representive) $\mathcal{Q} \in Quo(\mathcal{E}, \mathcal{F})$, let $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}} \subset \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ be the subfunctor whose S-points are given by the set of bundle maps $m : \mathcal{E}_S \to \mathcal{F}_S$ such that for every geometric point $x : \mathrm{Spa}(C', C'^{+}) \to S$, the subbundle $\mathrm{im}(x^* m : \mathcal{E}_{C'} \to \mathcal{F}_{C'}) \subset \mathcal{F}_{C'}$ is isomorphic to $\mathcal{Q}$.

Lemma 1.5. *For any $\mathcal{Q} \in Quo(\mathcal{E}, \mathcal{F})$, $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ is a (partially proper) locally spatial diamond. Moreover, $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ is either empty or equidimensional of dimension*

$$d(\mathcal{E}, \mathcal{F}; \mathcal{Q}) := \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

Note that $d(\mathcal{E}, \mathcal{F}; \mathcal{F}) = \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0}$, in accordance with the identification $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{F}} \cong \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$.

Proof Sketch. The functor $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ is a sub-v-sheaf of $\mathcal{H}\mathrm{om}$, hence a diamond by a theorem of Scholze. With a bit more work, one checks that $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ is locally spatial. By a Fitting ideal/constancy-of-HN-polygons argument, the image sheaf $\mathrm{im}(m : \mathcal{E}_S \to \mathcal{F}_S)$ associated with the map $m : \mathcal{E}_S \to \mathcal{F}_S$ parametrized by an S-point of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ defines a sub-vector bundle of $\mathcal{F}_S$, isomorphic to $\mathcal{Q}$ at all geometric points. Let $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}, \heartsuit}$ denote the $\mathrm{Aut}(\mathcal{Q})$-torsor over $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$ parametrizing isomorphisms $i : \mathcal{Q}_S \xrightarrow{\sim} \mathrm{im}\, m$. Then

$$\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}, \heartsuit} \cong \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{Q}) \times_{\mathrm{Spd}\, C} \mathcal{I}\mathrm{nj}(\mathcal{Q}, \mathcal{F})$$

is an open subdiamond of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{Q}) \times_{\mathrm{Spd}\, C} \mathcal{H}\mathrm{om}(\mathcal{Q}, \mathcal{F})$; moreover, the latter diamond is equidimensional of dimension

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0}.$$

Consequently, $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}, \heartsuit}$ is either empty or equidimensional of the same dimension. Now, since

$$f : \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}, \heartsuit} \to \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}}$$

is an $\mathcal{A}\mathrm{ut}(\mathcal{Q})$-torsor, every geometric fiber of f is (non-canonically) identified with $\mathcal{A}\mathrm{ut}(\mathcal{Q})$, so we'd like to argue that

$$\begin{aligned} \dim \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}, \heartsuit} - \dim \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})^{\mathcal{Q}} &= \dim \mathcal{A}\mathrm{ut}(\mathcal{Q}) \\ &= \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}. \end{aligned}$$

This follows from the next lemma, whose proof is omitted. ∎

Lemma 1.6. *Fix $C/\mathbf{F}_p$ complete algebraically closed, and let G be a locally spatial group diamond over $\mathrm{Spd}\, C$ whose identity component is geometrically smooth of dimension d. Then for $f : Y \to X$ any G-torsor in locally spatial diamonds over $\mathrm{Spa}\, C$, $\dim Y = \dim X + d$, and X is equidimensional if and only if Y is equidimensional.*

The idea now is that Theorem 1.4 follows *if* we can show that

$$d(\mathcal{E}, \mathcal{F}; \mathcal{Q}) < d(\mathcal{E}, \mathcal{F}; \mathcal{F}) = d_1$$

for every $\mathcal{Q} \in Quo(\mathcal{E}, \mathcal{F})$ with $\mathcal{Q} \subsetneq \mathcal{F}$, since Lemma 1.7 below shows that d_2 is the supremum of $d(\mathcal{E}, \mathcal{F}; \mathcal{Q})$ over such $\mathcal{Q}$'s. Writing out the definitions, this (putative) inequality becomes the inequality

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} < \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}. \quad (1)$$

Note that there is a possible hole/circularity in our argument here, since we just quantified over the set $Quo(\mathcal{E}, \mathcal{F})$ whose elements we don't understand how to enumerate! Let's work out some examples to see how one might get around this.

Example 1. As a warmup, let's examine the case where $\mathcal{F}$ is a line bundle. After twisting $\mathcal{E}$ and $\mathcal{F}$, we can assume without loss of generality that $\mathcal{F} \simeq \mathcal{O}$, so $Quo(\mathcal{E}, \mathcal{O})$ is a finite set of $\mathcal{O}(-j)$'s for some integers $j \geq 0$. In particular, for any given $\mathcal{Q} = \mathcal{O}(j)$, $\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} = j$ and $\deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0} = 0$. Finally, writing $\mathcal{E} \simeq \oplus \mathcal{O}(\lambda_i)$ for some $\lambda_1 \geq \lambda_2 \geq \ldots$, we have $\mathcal{E}^\vee \otimes \mathcal{Q} = \oplus \mathcal{O}(-\lambda_i - j)$, so

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} = \sum_{\lambda_i \leq -j} \deg \mathcal{O}(-\lambda_i - j) = \sum_{\lambda_i \leq -j} \deg \mathcal{O}(-\lambda_i) - j \cdot \mathrm{denom}(\lambda_i) =: c(j).$$

One easily checks that $c(j) < c(0)$ for every $j > 0$, so we get what we want: if $\mathrm{rank}(\mathcal{E}_{\leq \deg \mathcal{L}}) > 1$ then $\mathcal{L}$ is a quotient of $\mathcal{E}$.

Example 2. Let's check that $\mathcal{F} = \mathcal{O}(7/2)$ occurs as a quotient of $\mathcal{E} = \mathcal{O}(1/5)$. One easily computes that

$$d_1 = \dim \mathcal{H}\mathrm{om}(\mathcal{O}(1/5), \mathcal{O}(7/2)) = 33.$$

In this example, easy considerations with slope theory show that any $\mathcal{Q} \in Quo(\mathcal{E}, \mathcal{F})$ must be either $\mathcal{O}(7/2)$ or one of the following twelve bundles: $\mathcal{O}(1)$, $\mathcal{O}(2)$, $\mathcal{O}(3)$, $\mathcal{O}(1/2)$, $\mathcal{O}(3/2)$, $\mathcal{O}(5/2)$, $\mathcal{O}(i) \oplus \mathcal{O}(j)$ for any $3 \geq i \geq j \geq 1$. As $\mathcal{Q}$ varies over this previous list of bundles, the values of $d(\mathcal{E}, \mathcal{F}; \mathcal{Q}) = \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}$ are as follows:

- Take $\mathcal{Q} = \mathcal{O}(i)$, $1 \leq i \leq 3$. In this case $\deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0} = 0$ (which holds more generally for any semistable $\mathcal{Q}$), so dropping that term, we calculate

$$\begin{aligned} \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg \mathcal{O}(-\tfrac{1}{5}) \otimes \mathcal{O}(i) + \deg \mathcal{O}(\tfrac{7}{2}) \otimes \mathcal{O}(-i) \\ &= 9, 12, 15 \text{ for } i = 1, 2, 3 \text{ (respectively)}. \end{aligned}$$

- Take $\mathcal{Q} = \mathcal{O}(\frac{2i-1}{2})$ for $1 \leq i \leq 3$. Then

$$\begin{aligned} d(\mathcal{E}, \mathcal{F}; \mathcal{Q}) &= \deg \mathcal{O}(-\tfrac{1}{5}) \otimes \mathcal{O}(\tfrac{2i-1}{2}) + \deg \mathcal{O}(\tfrac{7}{2}) \otimes \mathcal{O}(\tfrac{1-2i}{2}) \\ &= (10i - 7) + (16 - 4i) \\ &= 9 + 6i \\ &= 15, 21, 27 \text{ for } i = 1, 2, 3 \text{ (respectively)}. \end{aligned}$$

- Take $\mathcal{Q} = \mathcal{O}(i) \oplus \mathcal{O}(j)$ for some $3 \geq i \geq j \geq 1$. Then one easily checks that $\deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0} = i - j$. Moreover,

$$\begin{aligned} \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} &= \deg \mathcal{O}(-\tfrac{1}{5}) \otimes \mathcal{O}(i) + \deg \mathcal{O}(-\tfrac{1}{5}) \otimes \mathcal{O}(j) \\ &= 5i + 5j - 2, \end{aligned}$$

  and

$$\begin{aligned} \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg \mathcal{O}(\tfrac{7}{2}) \otimes \mathcal{O}(-i) + \deg \mathcal{O}(\tfrac{7}{2}) \otimes \mathcal{O}(-j) \\ &= 14 - 2i - 2j. \end{aligned}$$

  Putting these together gives

$$d(\mathcal{E}, \mathcal{F}; \mathcal{Q}) = 2i + 4j + 12,$$

  and this is clearly maximized as a function of the relevant pairs (i, j) by taking $(i, j) = (3, 3)$, where its value is 30.

In summary, $\mathcal{H}\mathrm{om}(\mathcal{O}(1/5), \mathcal{O}(7/2))$ is a 33-dimensional diamond, while

$$\mathcal{H}\mathrm{om}(\mathcal{O}(1/5), \mathcal{O}(7/2)) \smallsetminus \mathcal{S}\mathrm{urj}(\mathcal{O}(1/5), \mathcal{O}(7/2))$$

is a 30-dimensional closed subdiamond. Therefore $\mathcal{S}\mathrm{urj}(\mathcal{O}(1/5), \mathcal{O}(7/2))$ is nonempty.

Note that in the previous examples we first enumerated some *possible* elements of $Quo(\mathcal{E}, \mathcal{F})$, i.e. a possibly larger list of "candidate elements" which weren't directly ruled out by slope-theoretic considerations, and then checked (1) for all $\mathcal{Q} \subsetneq \mathcal{F}$ in this set of bundles. This suggests the following outline for a proof of Theorem 1.4 in general:

Step One. Give a general precise definition of a set $Quo(\mathcal{E}, \mathcal{F})^?$ of "candidate elements of $Quo(\mathcal{E}, \mathcal{F})$". These should be defined purely by considerations of rank, degree, and HN polygons, in such a way that we have a priori inclusions $Quo(\mathcal{E}, \mathcal{F}) \subseteq Quo(\mathcal{E}, \mathcal{F})^?$ and $\mathcal{F} \in Quo(\mathcal{E}, \mathcal{F})^?$. (There is a natural candidate for $Quo(\mathcal{E}, \mathcal{F})^?$: one can take the set of bundles $\mathcal{Q}$ of rank $\leq \operatorname{rank} \mathcal{F}$ such that $\mathrm{HN}(\mathcal{Q}) \leq \mathrm{HN}(\mathcal{F})$ and $\mathrm{HN}(\mathcal{Q}^\vee) \leq \mathrm{HN}(\mathcal{E}^\vee)$. These conditions are certainly necessary.)

Step Two. Prove the strict inequality (1) for every $\mathcal{Q} \in Quo(\mathcal{E}, \mathcal{F})^?$ with $\mathcal{Q} \subsetneq \mathcal{F}$. This should "just" be combinatorics. Note, however, that if Step One isn't done correctly, then Step Two will definitely fail.

Step Three. Deduce Theorem 1.4 from the following easy lemma.

Lemma 1.7. *Let X be a locally spatial diamond, and let $\{X_i \subset X\}_{i \in I}$ be a finite collection of locally closed subdiamonds. Suppose that X and the X_i's are partially proper and that $|X| = \coprod_{i \in I} |X_i|$. Then $\dim X = \sup_{i \in I} \dim X_i$.*

Proof. By general nonsense, the X_i's are locally spatial, and each $|X_i|$ is stable under generalization inside $|X|$. Moreover, by the assumption of partial properness, each $|X_i|$ is also stable under specialization inside $|X|$. But then any chain of points inside

$$|X| = \coprod_{i \in I} |X_i|$$

necessarily lies in a single $|X_i|$, and the result follows. □

1.2 The semistable extension problem

Now we return to the motivating problem, in a special case. Given semistable bundles $\mathcal{E}_1, \mathcal{E}_2$ of degrees d_1, d_2, ranks r_1, r_2, and with slopes $\mu_1 = \frac{d_1}{r_1} < \mu_2 = \frac{d_2}{r_2}$, can we construct an extension

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

with $\mathcal{E}$ semistable? Here's one possible strategy.

Step One. Let $\mathcal{E}$ denote the (unique) semistable bundle of degree $d_1 + d_2$ and rank $r_1 + r_2$. Show that $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)$ is nonempty of the expected dimension $\deg(\mathcal{E}^\vee \otimes \mathcal{E}_2)^{\geq 0}$, which turns out to be $r_1 d_2 - r_2 d_1$.

Step Two. Stratify $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)$ into locally closed strata $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)_{\mathcal{Q}}$ where we've fixed the isomorphism type $\mathcal{Q}$ of $\ker \mathcal{E} \to \mathcal{E}_2$. (Any relevant $\mathcal{Q}$ has rank r_1 and degree d_1.) Note that $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)_{\mathcal{E}_1}$ is an open subdiamond of $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)$, by the usual semicontinuity business. We need to check that it is nonempty.

Step Three. Prove that for any relevant $\mathcal{Q}$ which is *not* semistable,

$$\dim \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2)_{\mathcal{Q}} < \dim \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{E}_2).$$

For this, observe that there is a natural $\mathcal{A}\mathrm{ut}(\mathcal{Q})$-torsor $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{E}_2)_{\mathcal{Q},\heartsuit}$ over $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{E}_2)_{\mathcal{Q}}$, which is naturally identified with a torsor (for the locally profinite group diamond $\mathcal{A}\mathrm{ut}(\mathcal{E}_2)$) over an open subdiamond of $\mathcal{S}\mathrm{urj}(\mathcal{E}^\vee,\mathcal{Q}^\vee)$. Arguing as before, deduce that $\mathcal{S}\mathrm{urj}(\mathcal{E},\mathcal{E}_2)_{\mathcal{Q}}$ (if nonempty) has dimension

$$\deg(\mathcal{Q}^\vee \otimes \mathcal{E})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

One is now reduced to checking that this quantity is strictly less than $r_1 d_2 - r_2 d_1$ for every non-semistable $\mathcal{Q}$ of rank r_1, degree d_1, and maximal HN slope $\leq \frac{d_1+d_2}{r_1+r_2}$.