

# LINE BUNDLES ON RIGID VARIETIES AND HODGE SYMMETRY

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ABSTRACT. We prove several related results on the low-degree Hodge numbers of proper smooth rigid analytic varieties over non-archimedean fields. Our arguments rely on known structure theorems for the relevant Picard varieties, together with recent advances in  $p$ -adic Hodge theory. We also define a rigid analytic Albanese naturally associated with any smooth proper rigid space.

## 1. INTRODUCTION

Let  $K$  be a  $p$ -adic field, i.e., a complete discretely valued extension of  $\mathbf{Q}_p$  with perfect residue field  $\kappa$ . Let  $X$  be a smooth proper rigid analytic space over  $K$ . In this paper, we study the relationship between the Hodge numbers  $h^{1,0}$  and  $h^{0,1}$  of  $X$ .

For a compact complex manifold  $Y$ , we always have  $\dim H^1(Y, \mathcal{O}_Y) \geq \dim H^0(Y, \Omega_Y^1)^{d=0}$  (c.f. [BHPVdV04, Chapter IV, Section 2]). In the rigid analytic setting, Scholze proved that the Hodge-de Rham spectral sequence always degenerates at  $E_1$ , and in particular every global 1-form on  $X$  as above is automatically closed (c.f. [Sch13, Theorem 8.4]). One is naturally led to guess that for  $X$  as above we always have  $\dim H^1(X, \mathcal{O}_X) \geq \dim H^0(X, \Omega_X^1)$ . In this paper we confirm this inequality assuming that  $X$  has a strictly semistable formal model (c.f. [HL00, Section 1]) and provide a geometric interpretation of the difference.

**Theorem 1.1** (Main Theorem). *Under the conditions stated above, we have*

$$\dim H^1(X, \mathcal{O}_X) \geq \dim H^0(X, \Omega_X^1).$$

*Moreover, the difference between two numbers above is the virtual torus rank of the Picard variety of  $X$  (defined in the next section).*

*Remark.*

- (1) It is true (although hard to prove) that possessing strictly semistable reduction is stable up under finite flat base extension, hence the theorem is insensitive to finite extensions of the ground field. Therefore we may and do assume that  $X$  has a  $K$ -rational point  $x : \mathrm{Sp}(K) \rightarrow X$ . We will fix this rational point from now on.
- (2) The proof relies crucially on the assumption that  $X$  has a strictly semistable formal model, which we use to determine the structure of the Picard variety of  $X$ , cf. Theorem 2.1 below. We certainly expect that the structure of the Picard variety should be of this shape in general. However, it is also a long standing folklore conjecture that any quasi-compact smooth rigid space potentially admits a strictly semistable formal model.
- (3) Assuming a result in progress by Conrad-Gabber along with the semistable reduction conjecture above, the Main Theorem holds for any smooth proper rigid space over any complete nonarchimedean extension of  $\mathbf{Q}_p$ .

In a complementary direction, the second author [Li17] singled out the class of smooth proper rigid spaces admitting some formal model with *projective* special fiber. In particular, according to Theorem 1.1 of loc. cit., the Picard variety of any such  $X$  is automatically proper. Combining this with Lütkebohmert's structure theorem (c.f. [Lüt95]) for proper rigid groups and the comparison results of [Sch13], we deduce the following result.

**Theorem 1.2.** *Let  $X$  be a smooth proper rigid space over a  $p$ -adic field  $K$ . Assume that  $X$  has a formal model  $\mathcal{X}$  over  $\mathrm{Spf}(\mathcal{O}_K)$  whose special fiber is projective. Then we have*

$$h^{1,0}(X) = h^{0,1}(X).$$

This result suggests that the condition of admitting a formal model with projective reduction could be a natural rigid analytic analogue of the Kähler condition. In particular, it is natural to ask if this condition implies Hodge symmetry in higher degrees:

**Question 1.3.** *Let  $X$  be a smooth proper rigid space admitting a formal model with projective reduction. Is it true that  $h^{i,j}(X) = h^{j,i}(X)$  for all  $i, j$ ?*

## 2. PRELIMINARIES

Throughout this section,  $X$  will be a smooth proper rigid space over a discretely valued non-archimedean field  $K$ .

In the paper [HL00], Lütkebohmert and Hartl considered the Picard functor

$$\underline{\text{Pic}}_{X/K} : (\text{Smooth Rigid Spaces over } K) \rightarrow (\text{Sets}), V \mapsto \underline{\text{Pic}}_{X/K}(V)$$

where

$$\underline{\text{Pic}}_{X/K}(V) = \{\text{Isomclass}(\mathcal{L}, \lambda) : \mathcal{L} \text{ a line bundle on } X \times_K V, \lambda : \mathcal{O}_V \xrightarrow{\sim} (x, \text{id})^* \mathcal{L} \text{ an isomorphism}\}.$$

Let us summarize several main statements of the paper mentioned above.

**Theorem 2.1** (Summary of Theorem 0.1, Proposition 3.13, Theorem 3.14 and Theorem 3.15 of [HL00]). *Assume  $X$  has a strictly semistable formal model.*

- (1) *The functor above is represented by a smooth rigid group denoted as  $\text{Pic}_X$ . After a suitable finite base extension we have:*
- (2) *The identity component  $\text{Pic}_X^0$  of  $\text{Pic}_X$ , i.e. the Picard variety of  $X$  in their terminology, canonically admits a Raynaud's uniformization:*

$$\begin{array}{ccccc} & & \Gamma & & \\ & & \downarrow & & \\ T & \longrightarrow & \hat{P} & \longrightarrow & B \\ & & \downarrow & & \\ & & \text{Pic}_X^0 & & \end{array}$$

Here  $T$  is a split torus of dimension  $r$ ,  $\Gamma$  is a lattice of rank  $k(\leq r)$ , and  $B$  is a good reduction abeloid variety (c.f. [Lüt95]), i.e.  $B$  is the rigid generic fiber of a formal abelian scheme.

- (3) *Non-canonically,  $\text{Pic}_X^0$  may be written as an extension of an abeloid variety by a split torus of dimension  $r - k$ :*

$$0 \rightarrow \mathbb{G}_m^{r-k} \rightarrow \text{Pic}_X^0 \rightarrow A \rightarrow 0.$$

- (4) *The geometric component group of  $\text{Pic}_X \times_K \overline{K}$ , i.e. the Néron-Severi group of  $X$  in their terminology, is a finitely generated abelian group.*

**Definition 2.2.** Define the virtual torus rank of  $\text{Pic}_X^0$  to be  $r - k$  in the notation above.

It is easy to derive the following structural properties of the Tate module of  $\text{Pic}_X$ .

**Proposition 2.3.**

- (1) *The Tate module of  $\text{Pic}_X$  is the same as that of  $\text{Pic}_X^0$ .*
- (2) *There are two canonical short exact sequences of  $p$ -adic  $G_K$  representations:*

$$0 \rightarrow V_p(T) \rightarrow V_p(\hat{P}) \rightarrow V_p(B) \rightarrow 0,$$

$$0 \rightarrow V_p(\hat{P}) \rightarrow V_p(\text{Pic}_X^0) = V_p(\text{Pic}_X) \rightarrow \varprojlim (\Gamma/p^n \Gamma) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow 0.$$

Here  $V_p(G) = \varprojlim G[p^n] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  denotes the rational  $p$ -adic Tate module associated with any commutative rigid analytic group  $G$ .

(3) *There is a non-canonical short exact sequence of  $p$ -adic  $G_K$  representations:*

$$0 \rightarrow \mathbb{Q}_p(1)^{r-k} \rightarrow V_p(\mathrm{Pic}_X^0) \rightarrow V_p(A) \rightarrow 0.$$

*Proof.* (1) follows from Theorem 2.1 (4), while (2) and (3) are consequences of Theorem 2.1 (2) and (3), respectively.  $\square$

### 3. PROOF OF THE MAIN THEOREM

With the aid of Proposition 2.3 and Hodge-Tate comparison, it is easy to prove the main theorem.

*Proof of Theorem 1.1.* By Hodge-Tate comparison for smooth proper rigid spaces over  $K$  (c.f. [Sch13, Theorem 7.11]), we have a canonical  $G_K$ -equivariant isomorphism

$$H_{\acute{e}t}^1(X_{\bar{K}}, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}_p} \mathbf{C} = (H^1(X, \mathcal{O}_X) \otimes_K \mathbf{C}(1)) \oplus (H^0(X, \Omega_X^1) \otimes_K \mathbf{C}),$$

where  $\mathbf{C}$  is the completion of an algebraic closure of  $K$ . On the other hand, by the usual Kummer sequence we have

$$H_{\acute{e}t}^1(X_{\bar{K}}, \mathbb{Q}_p(1)) = V_p(\mathrm{Pic}_X).$$

Combining these isomorphisms with Hodge-Tate comparison for  $A$  and the structural results for  $V_p(\mathrm{Pic}_X)$  described in Proposition 2.3 (3), we see that

$$\dim_K (V_p(\mathrm{Pic}_X) \otimes_{\mathbb{Q}_p} \mathbf{C})^{G_K} = \dim_K (V_p(A) \otimes_{\mathbb{Q}_p} \mathbf{C})^{G_K} = \dim_K (H^1(\hat{A}, \mathcal{O}_{\hat{A}})) = \dim(\hat{A}) = \dim(A)$$

and similarly

$$\dim_K (V_p(\mathrm{Pic}_X) \otimes_{\mathbb{Q}_p} \mathbf{C}(-1))^{G_K} = r - k + \dim(A),$$

where  $r - k$  is the virtual torus rank of the Picard variety of  $X$  as in §2.2. By Hodge-Tate comparison for  $X$ , the former is  $h^{1,0}(X)$  and the latter is  $h^{0,1}(X)$ , so taking the difference gives  $h^{0,1}(X) - h^{1,0}(X) = r - k$ , as desired.  $\square$

One sees that the argument above only uses the qualitative structure of the Picard variety. Similarly, is easy to prove Theorem 1.2.

*Proof of Theorem 1.2.* By work of Conrad-Gabber we may reduce ourselves to the situation where  $K$  is a  $p$ -adic field. Theorem [Li17, Theorem 1.1] says that in this situation the Picard variety is an abeloid variety. Therefore  $r - k = 0$ , so the argument above implies the equality  $h^{0,1}(X) = h^{1,0}(X)$ .  $\square$

### 4. THE ALBANESE

In this section we define another rigid group variety related to “1-motives of rigid spaces”, namely the rigid Albanese variety. We work in the slightly more general setting where  $X$  is a smooth proper rigid space over any complete nonarchimedean field  $K$  of characteristic 0; as before, we fix a rational point  $x : \mathrm{Sp}(K) \rightarrow X$ . The only non-formal input we require is the existence of the Picard variety associated with  $X$  in this generality, which is guaranteed by forthcoming work of Evan Warner, c.f. [War17].

**Definition 4.1.** The rigid Albanese variety  $(\mathcal{A}, 0)$  associated with  $(X, x)$  is the initial object in the category of pointed maps from  $(X, x)$  to an abeloid variety pointed at its origin.

If no confusion seems likely, we call  $\mathcal{A}$  the Albanese variety of  $X$  and denote it  $\mathrm{Alb}_X$ . If  $\mathrm{Alb}_X$  exists, it is clearly unique up to canonical isomorphism. In order to prove the existence of the Albanese, we employ the Picard variety as follows:

**Definition 4.2.** The Albanese  $\mathcal{A}$  of  $X$  is defined as the dual of the maximal connected proper smooth subgroup of the Picard variety of  $X$ .

Note that the maximal connected proper smooth subgroup of any commutative rigid analytic group is well-defined; this is an easy exercise which we leave to the reader.

**Proposition 4.3.** *The abeloid variety  $\mathcal{A}$  constructed above is the Albanese of  $X$ .*

*Proof.* To see that  $\mathcal{A}$  has the correct universal property, note that the Poincaré bundle on  $X \times \text{Pic}_X^0$  restricts to a line bundle on  $X \times \widehat{\mathcal{A}}$ . Therefore we have a morphism  $\text{Alb} : X \rightarrow \mathcal{A}$ . As the Poincaré bundle is trivialized along  $\{x\} \times \text{Pic}_X^0$ , we know that  $\text{Alb}(x) = 0$ . Now, given any pointed morphism  $\phi : (X, x) \rightarrow (\mathcal{A}', 0)$ , we may consider pullback of the Poincaré line bundle  $\mathcal{L}$  on  $\mathcal{A}' \times \widehat{\mathcal{A}'}$ .  $(\phi \times id_{\widehat{\mathcal{A}'}})^* \mathcal{L}$  on  $X \times \widehat{\mathcal{A}'}$  gives rise to a morphism  $\hat{\phi} : \widehat{\mathcal{A}'} \rightarrow \text{Pic}_X^0$ . This morphism necessarily factors through  $\hat{\phi} : \widehat{\mathcal{A}'} \rightarrow \widehat{\mathcal{A}}$  as  $\widehat{\mathcal{A}'}$  is proper and smooth. The dual of this morphism gives rise to a homomorphism  $\tilde{\phi} : \mathcal{A} \rightarrow \mathcal{A}'$ . Using functoriality of Picard and duality we see that  $\tilde{\phi}$  as constructed above is canonical and  $\phi = \tilde{\phi} \circ \text{Alb}$ . This completes the proof.  $\square$

The Albanese property implies that the induced map between first étale cohomology groups is injective.

**Proposition 4.4.** *For any prime  $l$ , the natural map  $\text{Alb}^* : H_{\text{ét}}^1(\text{Alb}_{X, \bar{K}}, \mathbb{Z}_l) \rightarrow H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Z}_l)$  is injective, and similarly for  $\mathbb{Q}_l$ -coefficients.*

*Proof.* It suffices to show the injectivity for  $\mathbb{F}_l$ -coefficients. An element  $\xi \in H_{\text{ét}}^1(\text{Alb}_{X, \bar{K}}, \mathbb{F}_l)$  is represented by an étale  $\mathbb{F}_l$ -torsor  $\mathcal{B}$  over  $\text{Alb}_{X, L}$  where  $L$  is a finite separable extension of  $K$ . In this situation  $\mathcal{B}$  itself is automatically an abeloid variety (c.f. [Mum08, P 167]<sup>1</sup>). Choose any class  $\xi$  such that  $\text{Alb}^* \xi = 0$ , in which case  $X \times_{\text{Alb}_X} \mathcal{B} = \mathcal{B}'$  is a trivial  $\mathbb{F}_l$ -torsor over  $X$  (possibly after passing to a finite extension of  $K$ ; from now on we will ignore the issue of base change and the reader should think of every statement as potentially true). In particular, we can choose a section  $\sigma : X \rightarrow \mathcal{B}'$  to the natural projection, as in the following diagram:

$$\begin{array}{ccc} \mathcal{B}' = X \times_{\text{Alb}_X} \mathcal{B} & \longrightarrow & \mathcal{B} \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \text{Alb}_X \end{array}$$

The section  $\sigma$  gives rise to a morphism from  $X$  to  $\mathcal{B}$  which can be chosen so that  $x$  is sent to 0. By the universal property of the Albanese we then get a section  $\tilde{\sigma} : \text{Alb}_X \rightarrow \mathcal{B}$ ; but this just means that  $\xi = 0$ , as desired.  $\square$

Parallel to the complex analytic situation, one might hope that the map from  $X$  to its Albanese could be given in terms of “integration of 1-forms on  $X$ ”. If  $K$  has positive characteristic, there is certainly no hope for this expectation due to pathological examples of algebraic varieties in characteristic  $p$ , so we suppose now that  $K$  is of characteristic 0. If the residue field  $\kappa$  is of characteristic 0 as well, then by [KKMSD73, P 198 Theorem II]  $X$  is potentially semistable. Therefore the discussion in Section 2 applies automatically; in particular, by Theorem 2.1 (3) we know that  $\widehat{\text{Alb}}_X$  has dimension no bigger than that of the abeloid variety  $A$  which appeared in the aforesaid Theorem.

On the other hand, if  $\kappa$  is of characteristic  $p$ , then we have no free access to a structure theorem for the Picard variety anymore. Nevertheless, if  $K$  is a  $p$ -adic field we can still prove that the dimension of the Albanese is no bigger than  $h^{1,0}(X)$  by combining Proposition 4.4 with a little  $p$ -adic Hodge theory.

**Proposition 4.5.** *If  $X$  is a smooth proper rigid space over a  $p$ -adic field  $K$ , then we have*

$$\dim(\text{Alb}_X) \leq \dim H^0(X, \Omega_X^1).$$

*Proof.* Proposition 4.4 implies that the dimension of the Hodge-Tate weight 1 (we follow the convention that  $\mathbb{Q}_p(1)$  has Hodge-Tate weight -1) piece of  $H_{\text{ét}}^1(\text{Alb}_{X, \bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{C}$  is smaller than that of  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{C}$ . One uses Hodge-Tate comparison again to see that the former is the dimension of  $\text{Alb}_X$  and the latter is  $\dim H^0(X, \Omega_X^1)$ .  $\square$

One might naïvely speculate that the dimension of  $\text{Alb}_X$  coincides with  $h^{1,0}(X)$ . However, we will see in the next section that this fails in general; see Example 5.6 for an explicit counterexample.

<sup>1</sup>The proof of the analogous result for abelian varieties given in loc. cit. extends with almost no change to the setting of abeloid varieties, except for the proof of the rigidity lemma. However, the proof of the rigidity lemma can be adapted: instead of arguing with Zariski closed subsets, one can use tubes of closed subsets of the special fiber of a formal model.

## 5. EXAMPLES

**Example 5.1.** Let  $A$  be an abeloid variety of dimension  $d$  over a discretely valued non-archimedean field. Its Picard variety is an abeloid variety of the same dimension  $d$ . Then we have  $h^{1,0} = h^{0,1} = d$  and the Albanese of  $A$  is of course  $A$  itself. The behavior of abeloid varieties is basically the same as abelian varieties according to [Lüt95].

**Example 5.2.** Let  $H$  be a non-archimedean Hopf variety (c.f. [Vos91] and [Mus77]). Then its Picard variety is  $\mathbb{G}_m$ . We have  $h^{1,0} = 0$  and  $h^{0,1} = 1$ . Since there is no non-constant morphism from a proper rigid variety to  $\mathbb{G}_m$  we see that the Albanese of  $H$  is trivial.

There is a geometric explanation for why Hopf varieties should have trivial Albanese.

**Proposition 5.3.** *Let  $K$  be a discretely valued non-archimedean field. There is no non-constant map from  $\mathbb{A}_K^{1,rig}$  to any Abeloid variety  $A$ .*

*Proof.* Applying [Lüt95, Main Theorem] (after possibly passing to a finite separable extension of  $K$ ) we may assume that  $A$  has a topological covering given by a smooth rigid group  $G$  sitting in an exact sequence of smooth rigid groups:

$$0 \rightarrow T \rightarrow G \rightarrow B \rightarrow 0.$$

Here  $T$  is a finite product of copies of  $\mathbb{G}_m^{rig}$ , and  $B$  is the generic fiber of a formal abelian scheme  $\mathcal{B}$  over  $\mathcal{O}_K$ . By [Lüt95, Section 2]  $\mathcal{B}$  is the Néron Model of  $B$ . In particular, for any admissible smooth formal scheme  $\mathcal{X}$  (whose generic fiber is denoted as  $X$ ) over  $\mathcal{O}_K$  and any morphism from  $X$  to  $B$ , it extends uniquely to a morphism from  $\mathcal{X}$  to  $\mathcal{B}$ . From [Ber90, Theorem 6.1.5] we see that  $\mathbb{A}^{1,rig}$  is topologically simply connected. So now every map  $\mathbb{A}^{1,rig} \rightarrow A$  would give rise to a map  $\mathbb{A}^{1,rig} \rightarrow G$ . It now suffices to show that any morphism from  $\mathbb{A}^{1,rig}$  to  $B$  or  $\mathbb{G}_m^{rig}$  must necessarily be constant. The latter being well known, we shall just prove the former.

We claim that any map  $f$  from  $\mathbb{A}^{1,rig}$  to  $B$  is trivial. To see this, choose an increasing nested sequence of closed discs  $D_i \subset \mathbb{A}^{1,rig}$  which admissibly cover  $\mathbb{A}^{1,rig}$ , and view  $f$  as the limit of its restrictions to the  $D_i$ 's. Now, closed discs have obvious smooth formal models with special fiber  $\mathbb{A}_\kappa^1$ . By the Néron mapping property, any map from a closed disc to  $B$  would extend to a map from such a smooth formal model to  $\mathcal{B}$ . Looking at the map on special fibers we get a map from a rational variety to an abelian variety, and any such map must be constant. Therefore our map  $f$  has image contained in an affinoid subspace of  $B$ . By the rigid analogue of Liouville's theorem (see Lemma 5.4)  $f$  must be constant.  $\square$

In this argument, we used the following (easy) rigid analytic analogue of Liouville's theorem.

**Lemma 5.4.** *There is no non-constant morphism from the analytification of a  $K$ -variety variety to a  $K$ -affinoid space.*

*Proof.* It suffices to prove the following:

*Claim.* Let  $X = \text{Spec } R$  be an affine integral scheme of finite type over  $K$ . Then every bounded analytic function on  $X^{rig}$  is a constant.

We achieve this in 2 steps.

Step 1: First, suppose that  $R = K[x_1, \dots, x_n]$ . We have to prove that every bounded analytic function on  $\mathbb{A}_K^{n,rig}$  is a constant. Recall that  $\mathbb{A}_K^{n,rig}$  is given by inductive limit of

$$\text{Sp } K\langle x_1, \dots, x_n \rangle \hookrightarrow \text{Sp } K\langle x_1, \dots, x_n \rangle \hookrightarrow \dots \hookrightarrow \text{Sp } K\langle x_1, \dots, x_n \rangle \hookrightarrow \dots,$$

so the set of analytic functions on  $\mathbb{A}_K^{n,rig}$  is given by

$$\bigcap_{k \in \mathbb{N}} K\langle p^k x_1, \dots, p^k x_n \rangle = \left\{ \sum_I a_I x^I \mid \lim_{I \rightarrow \infty} a_I p^{-k|I|} = 0, \forall k \in \mathbb{N} \right\}.$$

The boundedness of such a function translates to the existence of a constant  $C > 0$  such that

$$|a_I p^{-k|I|}| \leq C, \forall k \in \mathbb{N}, I = (i_1, \dots, i_n)$$

Therefore we get that each coefficient  $a_I$  must be zero except for  $I = (0, \dots, 0)$ , and our function is constant as desired.

Step 2: Choose an arbitrary  $R$  as in the claim. By Noether normalization, we may assume that  $R$  is a finite algebra over  $K[x_1, \dots, x_n]$ . We can even assume that  $R$  is the integral closure of  $K[x_1, \dots, x_n]$  in  $\text{Frac}(R)$  and that  $\text{Frac}(R)/\text{Frac}(K[x_1, \dots, x_n])$  is Galois with Galois group  $G$ . Let  $f$  be a bounded analytic function on  $X^{rig}$ , and consider the functions

$$a_i = \sum_{S \subset G, |S|=i} \prod_{g \in S} g(f)$$

where  $g(f)(x) = f(g(x))$ . It is easy to see that  $a_i$ 's are  $G$ -invariant, hence they are analytic functions on  $\mathbb{A}_K^{n,rig}$ . They are bounded functions, so by Step 1 they are constants. Since  $f$  satisfies the equation

$$f^n - a_1 f^{n-1} + \dots + (-1)^n a_n = 0,$$

we then see that  $f$  is a constant function as well.  $\square$

**Corollary 5.5.** *Let  $K$  be a discretely valued non-archimedean field. Let  $X$  be an  $\mathbb{A}^1$ -connected rigid variety. Then the Albanese of  $X$  is trivial.*

We illustrate the failure of “integrating 1-forms” through the following example.

**Example 5.6.** Let  $A$  be a simple abeloid variety of dimension  $d$  over a non-archimedean field  $K$ . Choose a non-torsion point  $P \in A$ . Let  $Y = (\mathbb{A}_K^2 - \{(0,0)\}) \times A$ . Consider a  $\mathbb{Z}$  action on  $Y$  given as dilation by some topologically nilpotent element on the first factor and translation by  $P$  on the second factor. This action is properly discontinuous. Take  $X = Y/\mathbb{Z}$ . Projection to the first factor makes  $X$  into an isotrivial family of abeloids over a Hopf surface  $H$ . One can compute the Hodge numbers of  $X$  via Leray spectral sequence applied to this projection. For example, we have  $h^{1,0}(X) = d$  and  $h^{0,1}(X) = d+1$ . But we make the following

*Claim.* The Albanese of  $X$  is trivial, i.e., there is no nontrivial map from  $X$  to any abeloid variety.

*Proof.* By our construction it suffices to show that there is no abeloid variety embedded as a subgroup of  $\text{Pic}_X^0$ . The fibration

$$\begin{array}{ccc} A & & \\ \downarrow & & \\ X & \longrightarrow & H \end{array}$$

gives an exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow \text{Pic}_X^0 \rightarrow \hat{A} \rightarrow 0$$

which exhibits  $\text{Pic}_X^0$  as the complement of the zero locus in the total space of a translation invariant line bundle  $\mathcal{L}$  on  $\hat{A}$ . Moreover, this translation invariant line bundle corresponds exactly to  $P \in \hat{A} = A$ . Now, a morphism from an abeloid variety  $\mathcal{A}$  to  $\text{Pic}_X^0$  is equivalent to the data of a homomorphism  $f : \mathcal{A} \rightarrow \hat{A}$  and an isomorphism  $s : \mathcal{O}_{\mathcal{A}} \rightarrow f^* \mathcal{L}$ . But since  $\hat{A}$  is simple and  $\mathcal{L}$  is non-torsion, such a morphism must be 0. Therefore we conclude that there is no nontrivial connected proper subgroup in  $\text{Pic}_X^0$ .  $\square$

An alternative argument due to Johan de Jong demonstrates that there is no non-constant morphism from  $Y$  (coming from  $X$ ) to an abeloid variety. Indeed, one notices that there is no non-constant morphism from a Hopf surface to an abeloid variety (c.f. Example 5.2). Therefore any morphism  $Y \rightarrow \mathcal{A}$  must factor through  $A$ . But since such a morphism comes from  $X$  hence it has to be invariant under translation by  $P$ . Thus we conclude that the Albanese of  $X$  above is trivial.

## 6. A LOOSE ENDING

In this section we would like to discuss several related problems left.

**6.1. A diagram.** Although the  $A$  in Theorem 2.1 (3) is non-canonical, we see that  $H^0(\hat{A}, \Omega_{\hat{A}}^1)$  is canonical in the sense that it is canonically isomorphic to  $H^0(X, \Omega_X^1)$  by  $p$ -adic Hodge theory. It would be appealing to get this just via coherent cohomology theory. To that end, let us consider  $c_{\text{dR}}^1(\mathcal{P}) \in H^1(X \times \text{Pic}_X^0, \Omega^1)$  where  $\mathcal{P}/X \times \text{Pic}_X^0$  is the Poincaré line bundle and  $c_{\text{dR}}^1$  is defined via

$$\text{dlog} : \mathcal{O}^* \rightarrow \Omega^1.$$

By Künneth formula we see that this gives rise to an element in  $H^0(X, \Omega_X^1) \otimes H^1(\text{Pic}_X^0, \mathcal{O})$  which we shall still denote as  $c_{\text{dR}}^1(\mathcal{P})$  for the sake of simplifying notations. Therefore we get a morphism

$$H^0(X, \Omega_X^1)^\vee \xrightarrow{c_{\text{dR}}^1(\mathcal{P})} H^1(\text{Pic}_X^0, \mathcal{O}).$$

Now there is also a Poincaré line bundle  $\mathcal{P}'$  over  $A \times \hat{A}$ . Pulling it back along  $\text{Pic}_X^0 \times \hat{A} \rightarrow A \times \hat{A}$  and apply the argument above, we get another morphism

$$H^0(\hat{A}, \Omega_{\hat{A}}^1)^\vee \xrightarrow{c_{\text{dR}}^1(\mathcal{P}')} H^1(\text{Pic}_X^0, \mathcal{O}).$$

We would like to make the following

**Conjecture 6.1.** *Both morphisms above are injective and have the same image.*

Here is a plausible strategy to prove this injectivity. Using the étale first Chern classes of  $\mathcal{P}$  and  $\mathcal{P}'$  we can get another pair of morphisms

$$H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))^\vee \xrightarrow{c_{\text{ét}}^1(\mathcal{P})} H_{\text{ét}}^1(\text{Pic}_X^0, \mathbb{Q}_p);$$

$$H_{\text{ét}}^1(\hat{A}_{\bar{K}}, \mathbb{Q}_p(1))^\vee \xrightarrow{c_{\text{ét}}^1(\mathcal{P}')} H_{\text{ét}}^1(\text{Pic}_X^0, \mathbb{Q}_p).$$

We believe these four morphisms are related by the following (imaginary) diagram:

$$\begin{array}{ccccc}
 & & H^1(\text{Pic}_X^0, \mathcal{O}) & \longleftarrow & \\
 & & \vdots \text{ ?} & & \swarrow c_{\text{dR}}^1(\mathcal{P}') \\
 & & H_{\text{ét}}^1(\text{Pic}_X^0, \mathbb{Q}_p) & \longleftarrow \supset & H_{\text{ét}}^1(A, \mathbb{Q}_p) \\
 & \nearrow c_{\text{dR}}^1(\mathcal{P}) & \uparrow c_{\text{ét}}^1(\mathcal{P}) & \longleftarrow c_{\text{ét}}^1(\mathcal{P}') & \parallel \\
 H^0(X, \Omega_X^1)^\vee & \cdots \longrightarrow & H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))^\vee & \longleftarrow & H_{\text{ét}}^1(\hat{A}_{\bar{K}}, \mathbb{Q}_p(1))^\vee \longleftarrow \cdots H^0(\hat{A}, \Omega_{\hat{A}}^1)^\vee \\
 & & \parallel & & \parallel \\
 & & V_p(\text{Pic}_X^0)^\vee & \longleftarrow & V_p(A)^\vee
 \end{array}$$

where a dotted arrow means that  $p$ -adic Hodge theory produces an actual arrow from the source to target *after* tensoring with  $\mathbf{C}$ . All the arrows except for the question marked one are the obvious ones. Unfortunately, we have no idea how to construct the question marked arrow. Perhaps one should use some feature of the pro-étale cover of  $\text{Pic}_X^0$ , namely the tower of iterated self multiplication by  $p$  maps (see also [Bha17, 2.2]). After constructing such an (imaginary) arrow and proving the commutativity of the diagram above, we can deduce the injectivity in Conjecture 6.1. However we have no idea of how to prove that they have the same image as the question marked arrow is probably not injective (even when the target is tensored with  $\mathbf{C}$ ).

**6.2. A little bit of  $p$ -adic Hodge theory.** When the base field is a  $p$ -adic field, one can calculate all the numerical invariants attached to  $\text{Pic}_X^0$  by looking at  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$ . We illustrate this by computing the dimension of  $B$ , which is canonically associated to  $\text{Pic}_X^0$  as in Theorem 2.1 (2), from the  $p$ -adic Hodge theoretic feature of  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$ . For simplicity, let us assume that  $X$  has a strictly semistable formal model over  $\mathcal{O}_K$ . Then  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$  will be a semistable representation. Now we have the following

**Lemma 6.2.** *Let  $V$  be a semistable  $p$ -adic Galois representation of  $G_K$  with Hodge-Tate weight valued in  $\{0, -1\}$ . Then there exists a canonical 2-step filtration  $0 \subseteq \text{Fil}^1 \subseteq \text{Fil}^2 \subseteq V$  on  $V$  such that all the graded pieces are crystalline representations. Moreover, the Hodge-Tate weight of  $V/\text{Fil}^2$  (respectively  $\text{Fil}^1$ ) is 0 (respectively  $-1$ ) and the nilpotent operator  $N$  induces an isomorphism*

$$N : V/\text{Fil}^2(1) \rightarrow \text{Fil}^1.$$

Again, we follow the convention that the Hodge-Tate weight of  $\mathbb{Q}_p(1)$  is  $-1$ .

This Lemma is certainly well-known to the experts, and we merely sketch the proof.

*Proof.* Consider the nilpotent operator  $N$  on  $\mathbb{D}_{\text{st}}(V)$ . Combining the weak admissibility of  $\mathbb{D}_{\text{st}}(V)$  with the limited range of Hodge-Tate weights, one calculates that  $N^2 = 0$  and that both  $\text{Image}(N)$  and  $\text{Ker}(N)$  are admissible  $(\phi, N)$  submodules in  $\mathbb{D}_{\text{st}}(V)$ . By a result of Colmez and Fontaine (c.f. [CF00, Théorème A]) we know that this gives rise to the filtration on  $V$  that we are looking for.  $\square$

Apply this Lemma we get a canonical 2-step filtration on  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$ . According to Proposition 2.3 (2), we also have a canonical 2-step filtration on  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$  given by  $V_p(T)$  and  $V_p(\hat{P})$ . These two filtrations are not the same because the first and third graded pieces of the second filtration do not have the same dimension in general. However, we believe these two filtrations are related in the following way.

**Conjecture 6.3.** *Let notation be as above. Then we have  $V_p(T) \subseteq \text{Fil}^1$  and  $V_p(\hat{P}) = \text{Fil}^2$ .*

This conjecture is motivated by [CI99] and provides a possible generalization of their result to rigid groups admitting Raynaud's presentation. Assuming this conjecture we can achieve the goal of this subsection.

**Proposition 6.4.** *Assuming Conjecture 6.3, the dimension of  $B$  is equal to the dimension of the weight 0 part of  $\text{Fil}^2 / \text{Fil}^1$  where  $\text{Fil}^i$  is the canonical filtration on  $H_{\text{ét}}^1(X_{\bar{K}}, \mathbb{Q}_p(1))$  described in Lemma 6.2.*

*Proof.* The conjecture indicates that the Hodge-Tate weights 0 and 1 part of the middle graded piece have the same dimension. From Proposition 2.3 (2) we see that this is the same as the dimension of  $B$ .  $\square$

**6.3. Equal characteristic 0.** We expect an analogous story for proper smooth rigid spaces over discretely valued equal characteristic 0 non-archimedean fields. Note that in this situation Theorem 2.1 applies for free. Although we do not have  $p$ -adic comparison theorems, one could hope to replace them by analogues of (mixed) Hodge theory. That is to say, one might be able to describe all the cohomology groups considered in this paper via cohomology and combinatorial data associated with a semistable reduction.

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