1 Smoothness of $\text{Bun}_n$ for dinguses\(^1\)

Dear Jared,

Let $\text{Bun}_n \to \text{Perf}$ denote the stack of rank $n$ vector bundles on “the” Fargues-Fontaine curve. Yesterday I figured out a fairly cheap argument for checking that $\text{Bun}_n$ is a smooth diamond stack, using charts made out of de Rham affine Grassmannians. Of course Peter’s charts made from those spaces $X_i$ give more information, but it seems harder to check that they have the right properties.

A word on terminology: if $S$ is any absolute diamond, we say $S$ is smooth if for any diamond $X$, the projection map $X \times S \to X$ is smooth. Note that if $S_1$ and $S_2$ are smooth, then so is $S_1 \times S_2$. One can also check that if $K$ is any finite extension of $\mathbb{Q}_p$ and $S$ is a diamond with a smooth morphism $S \to \text{Spd} K$, then $S$ is smooth in this sense.

Let $\text{Bun}^d_n \subset \text{Bun}_n$ denote the open-closed substack of bundles of constant degree $d$. Let $\text{Gr}_{n,k}/\text{Spd} \mathbb{Q}_p$ denote the de Rham affine Grassmannian sending $T \in \text{Perf}$ with specified untilt $T^\sharp$ to the set of subsheaves

$$\mathcal{E} \subset \mathcal{O}_{\mathcal{X}_T}^n$$

such that $\mathcal{E} \to \mathcal{O}_{\mathcal{X}_T}^n$ is a modification supported along $T^\sharp \subset \mathcal{X}_T$ of (constant) meromorphy type $(k,0,\ldots,0)$. Note that $\mathcal{E}$ has constant degree $-k$. In particular, for any $m \geq d/n$ there is a natural morphism

$$\text{Gr}_{n,mn-d} \to \text{Bun}_{n,d}$$

given by sending $\mathcal{E} \subset \mathcal{O}_{\mathcal{X}_T}^n$ as above to the degree $d$ bundle $\mathcal{E}(m) := \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(m)$. This clearly factors through a morphism

$$f_m: [\text{Gr}_{n,mn-d}/\text{GL}_n(\mathbb{Q}_p)] \to \text{Bun}_{n,d},$$

where $\text{GL}_n(\mathbb{Q}_p)$ acts on any $\text{Gr}_{n,k}$ in the usual way.

**Proposition 1.1.** The morphism $f_m$ is smooth.

*Proof.* We need to check that for any $S \in \text{Perf}$ and any morphism $a: S \to \text{Bun}_{n,d}$, the fiber product

$$S \times_{a, \text{Bun}_{n,d}, f_m} [\text{Gr}_{n,mn-d}/\text{GL}_n(\mathbb{Q}_p)]$$

is a diamond smooth over $S$. What functor does this fiber product represent? Well, giving $a$ is equivalent to giving a degree $d$ rank $n$ bundle $\mathcal{E}/\mathcal{X}_S$. Unwinding definitions then shows that this fiber product represents the set of isomorphism classes of pairs $(S^\sharp, \mathcal{E} \hookrightarrow \mathcal{F})$ where $S^\sharp$ is an untilt of $S$ and $\mathcal{E} \hookrightarrow \mathcal{F}$ is a modification supported along $S^\sharp \subset \mathcal{X}_S$ and of meromorphy type $(mn-d,0,\ldots,0)$, such that moreover $\mathcal{F}$ is pointwise-semistable.\(^2\) Ignoring the last condition, this functor is representable by a “twisted de Rham affine Grassmannian” $\text{Gr}_{n,d-mn/S}^\mathcal{E}$, which locally on $S$ is isomorphic to $\text{Gr}_{n,d-mn} \times S$ and therefore is smooth over $S$. Enforcing the semistability of $\mathcal{F}$ then cuts out (by Kedlaya-Liu) an open subspace

$$\text{Gr}_{n,d-mn}^{\mathcal{E},ss} \subset \text{Gr}_{n,d-mn}^\mathcal{E},$$

so $\text{Gr}_{n,d-mn}^{\mathcal{E},ss} \to S$ is still smooth, and

$$\text{Gr}_{n,d-mn}^{\mathcal{E},ss} \cong S \times_{\text{Bun}_{n,d}} [\text{Gr}_{n,mn-d}/\text{GL}_n(\mathbb{Q}_p)]$$

so we win.\(\square\)

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\(^2\)More precisely, this fiber product should be regarded as a functor on $\text{Perf}/S$, but whatever.
Next we describe the image of \( f_m \) on geometric points.

**Proposition 1.2.** Let \( C/\mathbb{F}_p \) be an algebraically closed perfectoid field, and let \( a : \text{Spd} \, C \to \text{Bun}_{n,d} \) be any point, with associated bundle \( E/X_C \). Then a lifts along \( f_m \) to a \( C \)-point of \( \text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p) \) if and only if the maximal Harder-Narasimhan slope of \( E \) is \( \leq m \).

**Proof.** “Only if” is an easy exercise: if \( a \) lifts, then by definition there is some inclusion \( E \leftarrow_{-m} O_{X_C} \), so \( E \leftarrow_{-m} O_{X_C} \) has maximal HN slope \( \leq 0 \). “If” can be deduced from various results of the form “weakly admissible filtrations of specified Hodge type on specified \( \varphi \)-modules exist when they should”.

The condition on HN slopes in the previous proposition cuts out an open substack \( \text{Bun}_{n,d}^{\leq m} \) such that \( f_m \) factors through the inclusion of this substack. Clearly \( \text{Bun}_{n,d}^{\leq m} \subset \text{Bun}_{n,d}^{\leq m+1} \) and

\[
\text{Bun}_{n,d} = \bigcup_{m \geq 0} \text{Bun}_{n,d}^{\leq m}.
\]

It is true, but not a priori obvious, that \( f_m : [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)] \to \text{Bun}_{n,d}^{\leq m} \) is surjective in the pro-\( \varepsilon \)-topology, i.e. that given any \( S \in \text{Perf} \) and any \( x \in \text{Bun}_{n,d}^{\leq m}(S) \) we can lift \( x \) along \( f_m \) after passing to some pro-\( \varepsilon \)-étale cover of \( S \). This can be deduced as follows: Using the previous two proposition, one first checks that the morphism of diamonds

\[
S \times_{s, \text{Bun}_{n,d}^{\leq m}, f_m} [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)] \to S
\]

is smooth, and moreover surjective on topological spaces, with locally spatial source. One then applies the following result (whose straightforward proof is omitted; the key point in the proof is that smooth maps of diamonds are universally open).

**Proposition 1.3.** Let \( f : Y \to X \) be any map of locally spatial diamonds. If \( f \) is smooth and \( |Y| \to |X| \) is surjective, then \( f \) is surjective as a map of pro-\( \varepsilon \)-étale sheaves.

OK, so we have a family of smooth maps

\[
f_m : [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)] \to \text{Bun}_{n,d}
\]

which together cover the target. Now comes the fun part.

**Proposition 1.4.** The stack \( [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)] \) is a smooth diamond stack.

With this in hand, we’re done: after choosing some smooth diamonds \( X_m \) with some smooth surjective maps

\[
g_m : X_m \to [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)],
\]

the composite maps \( f_m \circ g_m : X_m \to \text{Bun}_{n,d} \) are smooth and give a collection of charts which verify that \( \text{Bun}_{n,d} \) is a smooth diamond stack.

So now we need to show that \( [\text{Gr}_{n,mn-\varphi}/\text{GL}_n(\mathbb{Q}_p)] \) is smooth. We’d like to deduce this from the smoothness of \( \text{Gr}_{n,k} \). It turns out there’s a really cute general argument for this sort of thing (which is what I missed until yesterday).

**Proposition 1.5.** Fix a locally profinite group \( G \), and let \( X \) be any absolute diamond with \( G \)-action. If there exists some smooth diamond \( W \) with a free \( G \)-action, then \( [X/G] \) is a diamond stack. If moreover \( W \) can be chosen such that \( W[G] \) is smooth, then \( [X/G] \) is smooth whenever \( X \) is smooth.
Proof. Give \( X \times W \) the diagonal \( G \)-action; this action is free, since the action on \( W \) is free, so \((X \times W)/G\) is a diamond. The projection map \( X \times W \to X \) is smooth, surjective and \( G \)-equivariant, so we get a smooth surjective map

\[
(X \times W)/G \to [X/G]
\]

whose source is a diamond.\(^3\) Hence the target is a diamond stack.\(^4\)

Suppose now that \( X \) is smooth. The natural projection map \((X \times W)/G \to W/G\) is then smooth. Indeed, we get a pullback diagram

\[
\begin{array}{ccc}
X \times W & \to & W \\
\downarrow & & \downarrow \\
(X \times W)/G & \to & W/G
\end{array}
\]

with surjective pro-étale vertical maps, and smoothness of \( X \) implies that the upper horizontal map is smooth; since smoothness can be checked (quasi-)pro-étale-locally on the target, we get that the lower horizontal map is smooth as desired. But now, if \( W/G \) is smooth as well, we’re looking at a smooth map \((X \times W)/G \to W/G\) with smooth target, which implies that \((X \times W)/G\) is smooth. But then \((X \times W)/G \to [X/G]\) is a smooth surjective map whose source is a smooth diamond, so we win. \(\square\)

Returning to our specific situation, we just need to find some smooth diamond \( W \) with a free \( \text{GL}_n(\mathbb{Q}_p) \)-action, such that \( W/\text{GL}_n(\mathbb{Q}_p) \) is also smooth. To do this, suppose we can find smooth diamonds \( W_1 \) and \( W_2 \), where \( W_1 \) has a free \( \text{SL}_n(\mathbb{Q}_p) \)-action and \( W_2 \) has a free \( \mathbb{Q}_p^\times \)-action, such that \( W_1/\text{SL}_n(\mathbb{Q}_p) \) and \( W_2/\mathbb{Q}_p^\times \) are both smooth. Letting \( m : \text{SL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times \to \text{GL}_n(\mathbb{Q}_p) \) be the group homomorphism which is inclusion on the first factor and which sends \((1,a)\) to \( \text{diag}(a,\ldots,a) \), the diamond

\[
W = (W_1 \times W_2) \times_{\text{SL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times} \text{GL}_n(\mathbb{Q}_p)
\]

then does what we want: since \( \ker m \) is finite and \( \text{im } m \subset \text{GL}_n(\mathbb{Q}_p) \) is a finite-index normal subgroup, \( W \) is étale over the smooth diamond \( W_1 \times W_2 \), hence smooth itself, and

\[
W/\text{GL}_n(\mathbb{Q}_p) \cong W_1/\text{SL}_n(\mathbb{Q}_p) \times W_2/\mathbb{Q}_p^\times
\]

is smooth.

For \( W_2 \), we just take \( \text{Spd } \mathbb{Q}_p^\text{cyc} \cong \text{Spd } \mathbb{F}_p((t^{1/p^\infty})) \) with the usual \( \mathbb{Q}_p^\times \)-action. For \( W_1 \), it turns out that the following thing works. Let \( W_1 \) be the functor on \( \text{Perf} \) sending \( S \) to the set of pointwise-injective bundle maps \( i : \mathcal{O}^n \to \mathcal{O}(\frac{1}{n+1}) \) over the relative curve \( X_S \). There is an obvious \( \text{GL}_n(\mathbb{Q}_p) \)-action given by precomposition with \( i \). I claim that \( W_1 \) and \( W_1/\text{SL}_n(\mathbb{Q}_p) \) are smooth.\(^5\)

\(^3\)This follows from a general lemma: If \( P \) is some property of morphisms of diamonds which is stable under base change and quasi-pro-étale-local on the target, and \( Y \to X \) is a \( G \)-equivariant morphism of absolute diamonds which has \( P \), then \( [Y/G] \to [X/G] \) has \( P \), in the sense that for any diamond \( W \) with a map \( W \to [X/G], [Y/G] \times [X/G] \to W \) is a diamond and \( [Y/G] \times [X/G] \to W \) has \( P \).

\(^4\)One also checks that \([X/G]\) always has diagonal representable in diamonds, for any absolute diamond with \( G \)-action, cf. the “Notes on diamonds”.

\(^5\)It seems very likely that \( W_1/\text{GL}_n(\mathbb{Q}_p) \) is actually smooth, in which case one could avoid the silly circumlocutions of the previous paragraph, but I wasn’t able to see this smoothness immediately.
For the smoothness of $W_1$, consider the functor $W'$ on Perf sending $S$ to the set of sections $s \in H^0(\mathcal{X}_S, \mathcal{O}(\frac{1}{n+1}))$ such that $s$ does not vanish identically on any fiber of the map $|\mathcal{X}_S| \to |S|$. This functor is representable by a spatial diamond, which turns out by some games with Lubin-Tate formal modules to be of the shape $\text{Spd} \mathcal{F}_q((t^{1/p^\infty}))/\mathbb{Z}_{p^{n+1}}^\times$ for some free action of $\mathbb{Z}_{p}^\times \times \mathbb{Z}_{p^{n+1}}$ on some $\text{Spd} \mathcal{F}_q((t^{1/p^\infty}))$; in particular, this thing is smooth. (Here $\mathbb{Z}_{p}^h = \text{ring of integers in the degree $h$ unramified extension of } \mathbb{Q}_p$.)

Then $W_1$ is an open subfunctor of $W' \times \cdots \times W'$, so $W_1$ is smooth.

For the smoothness of $W_1/\text{SL}_n(\mathbb{Q}_p)$, we first observe that this thing has a moduli interpretation: it is the functor on Perf sending $S$ to the set of pairs $(\mathcal{E}, i)$ where $\mathcal{E} \subset \mathcal{O}(\frac{1}{n+1})/\mathcal{X}_S$ is a rank $n$ subbundle which is pointwise-semistable of degree zero and $i$ is a trivialization $i : \mathcal{O} \cong \wedge^n \mathcal{E}$. By some easy games with the classification, one can check that given any such $\mathcal{E}$, $\mathcal{O}(\frac{1}{n+1})/\mathcal{E}$ is a line bundle on $\mathcal{X}_S$ of constant degree 1, and that $i$ together with the trivialization $\mathcal{O}(1) \cong \wedge^{n+1} \mathcal{O}(\frac{1}{n+1})$ induce a canonical trivialization $\mathcal{O}(\frac{1}{n+1})/\mathcal{E} \cong \mathcal{O}(1)$. Pushing this further, $W_1/\text{SL}_n(\mathbb{Q}_p)$ identifies with the functor sending $S$ to the set of surjections $\mathcal{O}(\frac{1}{n+1}) \to \mathcal{O}(1)$ of bundles over $\mathcal{X}_S$; indeed, any such surjection has kernel $\mathcal{E}$ which is pointwise-semistable of degree zero and which comes with a canonical trivialization of its determinant, and then $W_1$ is the $\text{SL}_n(\mathbb{Q}_p)$-torsor over this guy parametrizing trivializations $\mathcal{O}^n \cong \mathcal{E}$ compatible with the trivialization of $\wedge^n \mathcal{E}$. Applying $\text{Hom}_{\mathcal{O}(\mathcal{X}_S)}(-, \mathcal{O}(1))$ to such a surjection gives an inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(\frac{n}{n+1})/\mathcal{X}_S$, nonzero on each fiber of the map $|\mathcal{X}_S| \to |S|$, with cokernel $\cong \mathcal{O}(1)^n$ at all geometric points of $S$. In particular, we get a natural transformation

$$f : W_1/\text{SL}_n(\mathbb{Q}_p) \to X$$

where $X$ is the functor sending $S$ to the set of sections $s \in H^0(\mathcal{X}_S, \mathcal{O}(\frac{n}{n+1}))$ which are not identically zero on any fiber of $|\mathcal{X}_S| \to |S|$. I claim that $f$ is an open immersion and that $X$ is smooth. For openness, one easily checks that $f$ is an injection. We then observe that $f$ identifies its source with the subfunctor of its target cut out by the requirement that the vector bundle $\mathcal{O}(\frac{n}{n+1})/\mathcal{O} \cdot s$ be pointwise-semistable, and the habitual openness of the latter condition gives what we want. Smoothness of $X$, finally, is analogous to the smoothness of $W'$ and is left as an exercise.

Cheers,

Dave