ON THE SUPERCUSPIDAL COHOMOLOGY OF BASIC LOCAL SHIMURA VARIETIES

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Abstract. We prove, under a mild condition, that the supercuspidal cohomology of basic local Shimura varieties is concentrated in the middle degree. The proof uses a mixture of local and global techniques, and relies crucially on the recent work of Fargues-Scholze. As a byproduct of our methods, we prove the cuspidal case of Fargues's geometrization conjecture for general linear groups, and deduce the strongest form of the Kottwitz conjecture for all basical local shtuka spaces associated with inner forms of GL_n.

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1. Introduction

1.1. **Background and main result.** Fix a reductive group $G/\mathbf{Q}_p$ and a conjugacy class of minuscule cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$. Let b be an element of the Kottwitz set $B(G,\mu)$, and let G_b the associated σ-centralizer group. Given such a datum (G,μ,b), Scholze constructed a tower of local Shimura varieties

$$\{\mathrm{Sh}(G,\mu,b)_K\}_{K\subset G(\mathbf{Q}_p)},$$

confirming conjectures of Rapoport-Viehmann [RV14, SW20]. This is a tower of smooth partially proper rigid spaces of dimension $d=\langle 2\rho,\mu\rangle$ over $\mathbf{C}_p$, generalizing the Rapoport-Zink spaces constructed in [RZ96]. The compactly supported ℓ-adic étale cohomology groups of this tower carry natural smooth commuting actions of $G(\mathbf{Q}_p)$ and $G_b(\mathbf{Q}_p)$, and also an action of the Weil group W_F where $F/\mathbf{Q}_p$ is the field of definition of μ.

It is a fundamental problem in the Langlands program to understand how these cohomology groups decompose representation-theoretically under the $G(\mathbf{Q}_p)\times G_b(\mathbf{Q}_p)\times W_F$-action. There are precise conjectures of Kottwitz and Harris-Viehmann which completely describe the alternating sum of the cohomologies in terms of putative local Langlands and local Jacquet-Langlands correspondences [RV14, §7-8]. However, even conjecturally, the individual cohomology groups are poorly understood.

In this paper, we make some progress towards understanding the individual cohomology groups of local Shimura varieties in arguably the most interesting case, namely when b is *basic*. To state our main result, we introduce some notation. Fix a prime $\ell\neq p$ and a finite extension $E/\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$. For any open compact subgroup $K\subset G(\mathbf{Q}_p)$, the ℓ-adic cohomology $R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E)$ (which requires some care to define correctly, cf. Definition 2.15) is a bounded complex of smooth $G_b(\mathbf{Q}_p)$-representations. Let ρ be any admissible smooth $G_b(\mathbf{Q}_p)$-representation on an E-vector space. Then we define

$$R\Gamma_c(G,\mu,b)[\rho] = \operatorname*{colim}_{K\to\{1\}} R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E) \otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))} \rho$$

and

$$H^i_c(G,\mu,b)[\rho] = H^i\left(R\Gamma_c(G,\mu,b)[\rho]\right).$$

Here $\mathcal{H}(G_b(\mathbf{Q}_p)) = C_c^\infty(G_b(\mathbf{Q}_p),E)$ is the usual smooth Hecke algebra. By the finiteness results proved in [FS21, §IX.3], $H^i_c(G,\mu,b)[\rho]$ is an admissible smooth $G(\mathbf{Q}_p)$-representation, which moreover has finite length if ρ has finite length, and $H^i_c(G,\mu,b)[\rho]=0$ unless $0\leq i\leq 2d$. Note that there is a natural edge map

$$\alpha : H^*_c(\mathrm{Sh}(G,\mu,b),E)\otimes_{\mathcal{H}(G_b(\mathbf{Q}_p))}\rho \to H^*_c(G,\mu,b)[\rho],$$

which is often (but not always) an isomorphism. In particular, one can check that α is an isomorphism if ρ is supercuspidal, so in this case $H^*_c(G,\mu,b)[\rho]$ coincides with the "naive" ρ-part of $H^*_c(\mathrm{Sh}(G,\mu,b),E)$.

It seems to be a folklore expectation that the most interesting part of $R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E)$ should be concentrated in the middle degree, see for instance [Ito13, p.115] and [RV14, Remark 7.4.(ii)]. This is of course in natural analogy with Arthur and Kottwitz's conjectures on the cohomology of global Shimura varieties [Art89, Kot90]. Our main result confirms a precise form of this expectation.

To state our main theorem, recall that given a reductive group $G/\mathbf{Q}_p$ with quasisplit inner form G^*, an L-parameter

$$\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell}) = \widehat{G}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p} \cong \widehat{G^*}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p}$$

is *supercuspidal* if it is semisimple and does not factor through a conjugate of ${}^LP(\overline{\mathbf{Q}_\ell}) = \widehat{P}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p}$ for any proper parabolic subgroup $P\subset G^*$. Such parameters are called discrete in [RV14], but this is arguably misleading.

Theorem 1.1. *Let (G,μ,b) be a basic local Shimura datum, and let ρ be a supercuspidal representation of $G_b(\mathbf{Q}_p)$. Suppose the following conditions hold.*

1. *The spaces $\mathrm{Sh}(G,\mu,b)_K$ occur in the basic uniformization at p of a global Shimura variety, in the sense of Definition 3.2.*
2. *The L-parameter $\varphi_\rho : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ associated with ρ by Fargues-Scholze* [FS21, §1.9] *is supercuspidal.*

Then $H^i_c(G,\mu,b)[\rho]=0$ for all $i\neq d=\dim \mathrm{Sh}(G,\mu,b)_K$.

To the best of our knowledge, this is the first general vanishing theorem for the cohomology of local Shimura varieties.

Let us comment on the conditions in this theorem. Condition 1. is absolutely essential for our proof, although we believe the theorem holds without this assumption. Nevertheless, it is plausible that condition 1. holds for every basic local Shimura datum. With current technology, condition 1. can be verified for many (but not all) basic local Shimura data, including most cases of classical interest, cf. §3.1.

Condition 2. may appear strange at first sight, but in reality it is natural. In fact, this condition is already necessary in the case of the Lubin-Tate tower. Recall that in this situation, $G=\mathrm{GL}_{n/\mathbf{Q}_p}$ and $G_b(\mathbf{Q}_p) = D^\times$ where $D/\mathbf{Q}_p$ is the division algebra of invariant $1/n$. Then $G_b(\mathbf{Q}_p)$ is compact modulo center, and any irreducible ρ is supercuspidal (in a somewhat vacuous sense). However, for any such ρ whose Jacquet-Langlands transfer to GL_n is not supercuspidal, Boyer proved that the groups $H^i(G,\mu,b)[\rho]$ are *not* concentrated in the middle degree [Boy09]. This shows that *some* condition on ρ beyond mere supercuspidality is necessary for the conclusion of Theorem 1.1 to hold. However, for inner forms of GL_n, the Fargues-Scholze construction of L-parameters agrees with the usual local Langlands correspondence by [FS21, Theorem IX.7.4] and [HKW22, Theorem 1.0.3], and these bad ρ's are exactly the $G_b(\mathbf{Q}_p)$-representations which violate condition 2.

For a more subtle example, consider the basic local Shimura datum (G,μ,b) where G is the unique nonsplit inner form of $\mathrm{GSp}_4/\mathbf{Q}_p$ and μ is the Siegel cocharacter. Note that in this case $G_b(\mathbf{Q}_p)\simeq \mathrm{GSp}_4(\mathbf{Q}_p)$ is a split group. Then for certain supercuspidal ρ, Ito-Mieda have shown that the cohomology $H^i_c(G,\mu,b)[\rho]$ is not concentrated in the middle degree [Mie21]. However, the ρ's examined by Ito-Mieda lie in a non-tempered

A-packet, and condition 2. is again known to fail for these particular choices of ρ.[1] More generally, if one believes that the Fargues-Scholze construction realizes the "true" local Langlands correspondence, then condition 2. morally amounts to the requirement that for any pure inner form H of G and any L-packet Π for H containing some generalized Jacquet-Langlands transfer of ρ, the packet Π is supercuspidal.

We now give a detailed sketch of the proof of Theorem 1.1. In most of the following discussion, we will ignore two major technical complications, which we will highlight at the very end.

Condition 1. implies, more or less by definition, that we can choose a global Shimura datum $(\mathbf{G}, X)$ such that the basic Newton strata in the associated Shimura varieties are uniformized by the local Shimura varieties $\mathrm{Sh}(G, \mu, b)_K$. Moreover, for such a datum, $\mathbf{G}$ admits a canonical inner form such that $\mathbf{G}'_{\mathbf{Q}_p} \simeq G_b$, $\mathbf{G}'_{\mathbf{A}^{f,p}} \simeq \mathbf{G}_{\mathbf{A}^{f,p}}$, and $\mathbf{G}'(\mathbf{R})$ is compact modulo center (Proposition 3.1). By a standard argument with the simple trace formula, we can choose a compact open subgroup $K^p \subset \mathbf{G}'(\mathbf{A}^{f,p})$ and an algebraic $\mathbf{G}'$-representation $\mathcal{L}_\xi$ of some regular highest weight such that possibly after replacing ρ by an unramified twist, ρ occurs as a direct summand of a suitable space $\Pi = \mathcal{A}(\mathbf{G}'(\mathbf{Q})\backslash \mathbf{G}'(\mathbf{A}^f)/K^p, \mathcal{L}_{\xi,E})$ of algebraic automorphic forms with infinite level at p.

From the geometry of the uniformization isomorphism, we easily obtain a $G(\mathbf{Q}_p)$-equivariant map

$$R\Gamma_c(\mathrm{Sh}(G, \mu, b), E) \otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))} \Pi \overset{\Theta}{\to} R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E}).$$

The idea that uniformization of the basic locus induces a map like this is certainly not new, and goes back in various guises to work of Carayol, Harris-Taylor, Fargues and Mantovan [Car90, Far04, HT01, Man04]. The key new idea in our proof is to show that for any supercuspidal ρ occurring as a direct summand of Π which moreover satisfies condition 2. in Theorem 1.1, the induced map

$$R\Gamma_c(G, \mu, b)[\rho] \overset{\Theta_\rho}{\to} R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E})$$

realizes the left-hand side as a $G(\mathbf{Q}_p)$-stable *direct summand* of the right-hand side. If the global Shimura varieties are compact, Theorem 1.1 now follows immediately, since $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E})$ is concentrated in the middle degree by Matsushima's theorem together with standard vanishing properties of $(\mathfrak{g}, K_\infty)$-cohomology in regular weights. In the general case, we use an extra duality argument (Theorem 2.23) and a more general vanishing theorem due to Li-Schwermer [LS04].

Perhaps surprisingly, our argument that Θ_ρ is a split inclusion doesn't require any particular knowledge about the globalization of ρ to an automorphic form on $\mathbf{G}'$. We also don't need any information about non-basic Newton strata in the global Shimura variety, not even their existence. The key idea instead is to split Θ_ρ by splitting a suitable map of sheaves on a flag variety which induces the map Θ_ρ on global sections. Here we make crucial use of condition 2., the geometry of the Hodge-Tate period map, and the relationship between representation theory and sheaves on the stack Bun_G of G-bundles on the Fargues-Fontaine curve.

Very roughly, the idea is to look at the Hodge-Tate period map $\pi_{\mathrm{HT}} : \mathrm{Sh}(\mathbf{G}, X)_{K^p} \to \mathscr{F}\ell_{G,\mu}$. As in [CS17], this flag variety admits a Newton stratification, and the stratum associated with the basic element $b \in B(G, \mu)$ defines an open $G(\mathbf{Q}_p)$-stable subset $\mathscr{F}\ell^b_{G,\mu} \overset{j}{\to} \mathscr{F}\ell_{G,\mu}$ which moreover admits a $G(\mathbf{Q}_p)$-equivariant map to the classifying stack $B\underline{G_b(\mathbf{Q}_p)}$. In particular, given any ρ, one can produce an equivariant sheaf $j_!\mathcal{F}_\rho$ on $\mathscr{F}\ell_{G,\mu}$ such that $R\Gamma_c(G, \mu, b)[\rho] \cong R\Gamma(\mathscr{F}\ell_{G,\mu}, j_!\mathcal{F}_\rho)$. On the other hand, the map Θ arises as the global sections of a map

$$j_!\mathcal{F}_\Pi \cong j_!j^*R\pi_{\mathrm{HT}*}\mathcal{L}_{\xi,E} \to R\pi_{\mathrm{HT}*}\mathcal{L}_{\xi,E},$$

where the identification $\mathcal{F}_\Pi \cong j^*R\pi_{\mathrm{HT}*}\mathcal{L}_{\xi,E}$ is a direct consequence of the uniformization isomorphism. Our chosen globalization of ρ gives rise to a map $j_!\mathcal{F}_\rho \to j_!\mathcal{F}_\Pi$ such that Θ_ρ arises as the global sections of the composite map

$$\theta_\rho : j_!\mathcal{F}_\rho \to j_!\mathcal{F}_\Pi \to R\pi_{\mathrm{HT}*}\mathcal{L}_{\xi,E},$$

so it now suffices to split the map θ_ρ.

[1] More precisely, the Langlands parameter associated with ρ by the GSp_4 local Langlands correspondence of Gan-Takeda [GT11] is not supercuspidal, and Hamann has recently proved [Ham21] that Fargues-Scholze parameters for $\mathrm{GSp}_4/\mathbf{Q}_p$ coincide with Gan-Takeda parameters (at least if $p > 2$).

To split this map, the essential new ingredient is the following notion (which we state in a slightly incorrect way in this introduction, cf. Definition 2.21 for the true definition).

Definition 1.2. Let $b \in B(G)_{\mathrm{bas}}$ be any basic element, and let $h : \underline{BG_b(\mathbf{Q}_p)} \cong \mathrm{Bun}_G^b \to \mathrm{Bun}_G$ be the associated open immersion. An irreducible smooth $G_b(\mathbf{Q}_p)$-representation ρ is *inert* if some twist of ρ contains an invariant $\mathcal{O}$-lattice, and the associated ℓ-adic sheaf $\mathcal{F}_\rho$ on $\underline{BG_b(\mathbf{Q}_p)}$ satisfies $h_!\mathcal{F}_\rho \xrightarrow{\sim} Rh_*\mathcal{F}_\rho$.

By some straightforward arguments, the subcategory of $\mathrm{Rep}_E(G_b(\mathbf{Q}_p))$ spanned by inert representations is stable under twisting and contragredients. However, the following result lies significantly deeper.

Theorem 1.3. *Let $\rho \in \mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ be any irreducible smooth representation. If the associated L-parameter φ_ρ is supercuspidal, then ρ is supercuspidal and inert.*

This theorem ultimately follows from the compatibility between the Fargues-Scholze construction of Langlands parameters and parabolic induction, and relies on the full power of the machinery developed in [FS21]. Theorem 1.3 is a powerful result, and it is the key technical input into our argument. With this result in hand, we split θ_ρ by a surprising argument with the six functor formalism (Lemma 3.8), using crucially that $j_!\mathcal{F}_\rho$ arises via pullback from a sheaf on Bun_G.

It is instructive to consider the statement and proof of Theorem 1.1 in the most classical and well-understood situations, namely for the Lubin-Tate and Drinfeld local Shimura data $(\mathrm{GL}_n, (z, 1, ..., 1), b)$ and $(D^\times, (z^{-1}, 1, ..., 1), b^{-1})$. Recall that for these data we have $G_b = D^\times$ and GL_n, respectively. Then condition 1. holds (up to a harmless $\mathbf{G}_m$ factor) with $(\mathbf{G}, X)$ a Harris-Taylor Shimura datum. In the Drinfeld case, condition 2. always holds, and in the Lubin-Tate case it reduces (as discussed above) to the assumption that ρ is the Jacquet-Langlands transfer of a supercuspidal representation of GL_n. In the Drinfeld situation, Harris proved Theorem 1.1 by a global argument [Har97]. Faltings then proved the Lubin-Tate case of Theorem 1.1, also by a global argument, and Mieda later gave a purely local proof [Fal02, Mie10]. However, these situations are extremely special: in the Drinfeld case, the map Θ is an *isomorphism*, while in the Lubin-Tate case a soft argument ("Boyer's trick") shows that it becomes an isomorphism after passing to supercuspidal parts for the $G(\mathbf{Q}_p)$-action. In general, the map Θ is certainly not an isomorphism on supercuspidal parts, and our argument for the splitting of θ_ρ goes far beyond Boyer's trick.

In this sketch above, we have ignored two key technical issues. The first issue is that we have pretended to use a six functor formalism for étale cohomology of diamonds with E-coefficients with many good properties: excision, smooth and proper base change, comparison with $D_c^b(X, E)$ for schemes, comparison with $D_{\mathrm{lis}}(\mathrm{Bun}_G, E)$, etc. Unfortunately, such a formalism does not exist. This forces us to work with $\mathcal{O}$-integral objects everywhere in the proof, so that we can use the six functor formalism developed in [Sch17, §26]. Since representation theory with $\mathcal{O}$-coefficients is more subtle than with E-coefficients, this leads to some real complications in the representation-theoretic portions of the proof. On the other hand, we are allowed to systematically ignore (bounded) $\mathcal{O}$-torsion throughout the arguments, since we will invert ℓ at the end; we develop some simple language and techniques to facilitate this in §2.1.

The second issue is that in order to apply the critical Lemma 3.8, a certain map towards Bun_G must be ℓ-cohomologically smooth. This forces us to take quotients by an open compact subgroup $K_p \subset G(\mathbf{Q}_p)$ everywhere in the argument above, and then shrink K_p at the very last step. The essential point here is that the Beauville-Laszlo map $\mathscr{F}\ell_{G,\mu} \to \mathrm{Bun}_G$ is not ℓ-cohomologically smooth, but it is the inverse limit of the system of maps $[\mathscr{F}\ell_{G,\mu}/\underline{K_p}] \to \mathrm{Bun}_G$, each of which is ℓ-cohomologically smooth (Proposition 2.10).

Finally, we note that our argument shows that Theorem 1.1 holds with condition 2. replaced by the apparently more general condition that ρ is supercuspidal and inert. However, Scholze has conjectured that the converse of Theorem 1.3 is true, so this extra generality is probably illusory.

1.2. **Corollaries.** Our main theorem has a number of consequences.

First of all, combining Theorem 1.1 with [HKW22, Theorem 1.0.2] leads to the following result.

Theorem 1.4. *Choose any (G, μ, b) and ρ satisfying the conditions of Theorem 1.1, with $d = \dim \mathrm{Sh}(G, \mu, b)_K$ as before. Suppose moreover that the refined local Langlands correspondence* [Kal16, Conjecture G] *holds for G and its extended pure inner forms. Let ϕ be the L-parameter associated with ρ as in* [Kal16, Conjecture

G], *and assume that every member of the L-packet* $\Pi_\phi(G)$ *is supercuspidal. Then as* $G(\mathbf{Q}_p)$*-representations, we have*

$$H^i_c(G,\mu,b)[\rho] \simeq \begin{cases} \oplus_{\pi\in\Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)\cdot\pi & \text{if } i=d \\ 0 & \text{if } i\neq d. \end{cases}$$

Here r_μ *is the natural representation of* $\widehat{G}\rtimes W_F$ *extending the highest weight representation* r_μ *of* $\widehat{G}$*, and the remaining notation follows* [HKW22] *and* [Kal16].

In other words, conditionally and ignoring the W_F-action, we confirm a strong form of the Kottwitz conjecture, without passing to the Grothendieck group. Note that here we commit the usual venal sin of referring to **the** local Langlands correspondence, despite the fact that a uniquely determined local Langlands correspondence with all expected properties is not yet known for general groups, nor is there even a known list of conditions which would uniquely determine such a correspondence (but see [BMY20] for some recent progress towards this). Of course, it is completely natural to expect that any such correspondence will be compatible with the construction of Fargues-Scholze. In particular, it is natural to expect (in the notation of the previous theorem) that $\phi=\varphi_\rho$, but amusingly we don't need to impose any compatibility between ϕ and φ_ρ.

Next we specialize our results to the case of local Shimura data of unramified EL type. These data can be described very concretely. Precisely, we take $G=\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite unramified extension $L/\mathbf{Q}_p$. Writing $\Sigma=\mathrm{Hom}(L,\overline{\mathbf{Q}_p})$, we then choose any

$$\mu=\prod_{\sigma\in\Sigma}\mu_\sigma : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}} \cong \prod_{\sigma\in\Sigma}\mathrm{GL}_{n,\overline{\mathbf{Q}_p}}$$

where $\mu_\sigma(z)=(\underbrace{z,...,z}_{d_\sigma},\underbrace{1,...,1}_{n-d_\sigma})$ for some arbitrary integers $0\le d_\sigma\le n$. When Σ contains an element τ such that

$$d_\sigma\in\begin{cases}\{1,n-1\} & \text{if } \sigma=\tau\\ \{0,n\} & \text{if } \sigma\neq\tau,\end{cases}$$

the tower $\mathrm{Sh}(G,\mu,b)_K$ recovers the classical Lubin-Tate tower for $\mathrm{GL}_{n/L}$. However, for any other choice of μ, the geometry of the spaces $\mathrm{Sh}(G,\mu,b)_K$ is still very mysterious. Note that in general, $\dim\mathrm{Sh}(G,\mu,b)_K = \sum_\sigma d_\sigma(n-d_\sigma)$.

Now, by [FS21, Theorem IX.7.4] and [HKW22, Theorem 1.0.3], for any inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ the Fargues-Scholze construction provably recovers the usual semisimplified local Langlands correspondence [HT01]. In particular, $\rho\in\mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ satisfies condition 2. in Theorem 1.1 if and only if the Jacquet-Langlands transfer of ρ to $\mathrm{GL}_n(L)$ is supercuspidal. Combining these observations and Theorem 1.1 with previous works of Fargues and Shin on Kottwitz's conjecture in the unramified EL case, cf. [Far04, Shi12b], we deduce the following result.

Theorem 1.5. *Let* (G,μ,b) *be a basic local Shimura datum of unramified EL type, with* $d=\dim\mathrm{Sh}(G,\mu,b)_K$*. Let* $\pi\in\mathrm{Irr}_E(G(\mathbf{Q}_p))$ *be any supercuspidal representation, with Jacquet-Langlands transfer* $\mathrm{JL}(\pi)\in\mathrm{Irr}_E(G_b(\mathbf{Q}_p))$*. Then as* $G(\mathbf{Q}_p)\times W_F$*-representations, we have*

$$H^i_c(G,\mu,b)[\mathrm{JL}(\pi)] \simeq \begin{cases} \pi\boxtimes(r_\mu\circ\varphi_\pi|_{W_F}\otimes|\cdot|^{-d/2}) & \text{if } i=d\\ 0 & \text{if } i\neq d.\end{cases}$$

Here $\varphi_\pi : W_{\mathbf{Q}_p}\to {}^LG(\overline{\mathbf{Q}_\ell})=\widehat{G}(\overline{\mathbf{Q}_\ell})\rtimes W_{\mathbf{Q}_p}$ *is the L-parameter of* π*, and* r_μ *is the natural representation of* $\widehat{G}\rtimes W_F$ *extending the highest weight representation* r_μ *of* $\widehat{G}$.

This confirms the strongest possible form of the Kottwitz conjecture for local Shimura varieties of unramified EL type. To the best of the author's knowledge, prior to this paper, the Lubin-Tate and Drinfeld towers were (up to trivialities) the only basic local Shimura towers for which this optimal result was known.

We also get the following unconditional cases of Theorem 1.4.

Theorem 1.6. *Let (G, μ, b) be any basic local Shimura datum such that G is an inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite extension $L/\mathbf{Q}_p$. Let $\rho \in \mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ be any irreducible representation whose Jacquet-Langlands transfer to $\mathrm{GL}_n(L)$ is supercuspidal. Then*

$$H^i_c(G, \mu, b)[\rho] \simeq \begin{cases} \dim r_\mu \cdot \mathrm{JL}(\rho) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Here $\mathrm{JL}(\rho)$ is the generalized Jacquet-Langlands transfer of ρ to $G(\mathbf{Q}_p)$ [DKV84, Rog83].

This notably requires some cases of basic uniformization for Shimura varieties with bad reduction at p, cf. Theorem 3.5.

To explain our next theorem, let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ be the triangulated category constructed in [FS21, §VII.7]. If $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ is any Langlands parameter, the machinery in [FS21, §IX.7] defines a full triangulated subcategory $\mathscr{C}_\varphi \subset D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$, which is roughly the subcategory of "φ-isotypic objects" for the action of the algebra of excursion operators. In general, $\mathscr{C}_\varphi$ is stable under the Hecke action of $\mathrm{Rep}(\widehat{G})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$. If φ is supercuspidal, one can also show that any object of $\mathscr{C}_\varphi$ is !-extended from the open substack $\mathrm{Bun}_G^{\mathrm{ss}}$; in other words, $\mathscr{C}_\varphi$ is naturally contained in the full subcategory

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathrm{ss}}, \overline{\mathbf{Q}_\ell}) = \prod_{b \in B(G)_{\mathrm{bas}}} D(G_b(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell}),$$

where $D(G, \overline{\mathbf{Q}_\ell})$ denotes the derived category of smooth $\overline{\mathbf{Q}_\ell}$-representations of G. In particular, when φ is supercuspidal, the category $\mathscr{C}_\varphi$ carries a natural t-structure, by restriction from the standard t-structures on the categories $D(G_b(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell})$. Moreover, the heart of this t-structure is a full subcategory of

$$\prod_{b \in B(G)_{\mathrm{bas}}} \mathrm{Rep}_{\overline{\mathbf{Q}_\ell}}(G_b(\mathbf{Q}_p)),$$

so this heart is a very concrete category of representation-theoretic nature.

In his 2017 IHES lectures, Scholze proposed the following.

Conjecture 1.7 (Scholze)**.** *If $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ is any supercuspidal L-parameter, the Hecke action of $\mathrm{Rep}(\widehat{G})$ on $\mathscr{C}_\varphi$ is t-exact for the natural t-structure. In particular, it restricts to a natural Hecke action on $\mathscr{C}_\varphi^\heartsuit$.*

Our next theorem confirms Scholze's t-exactness conjecture, as well as the supercuspidal case of Fargues's geometrization conjecture [Far16b] for general linear groups.

Theorem 1.8. *Let G be a restriction of scalars of GL_n, and let $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ be any supercuspidal L-parameter. Then Conjecture 1.7 is true. Moreover, $\mathscr{C}_\varphi^\heartsuit$ admits a canonical generator $\mathscr{F}_\varphi$.*

The sheaf $\mathscr{F}_\varphi$ is an irreducible Hecke eigensheaf with eigenvalue φ, and its stalk at $B\underline{G_d(\mathbf{Q}_p)}$ is isomorphic to $\mathcal{F}_{\pi_{\varphi,d}}$ for all $d \in \mathbf{Z} \cong B(G)_{\mathrm{bas}}$.

Here we change notation slightly, and write $G_d = G_{b_d}$ for the inner form associated with the isoclinic isocrystal of slope d/n. Then $\pi_{\varphi,d}$ is the unique irreducible representation of $G_d(\mathbf{Q}_p)$ with L-parameter φ, and $\mathcal{F}_{\pi_{\varphi,d}}$ is the associated sheaf on $B\underline{G_d(\mathbf{Q}_p)}$.

The proof of Theorem 1.8 combines Theorem 1.6 with some recent work of Anschütz and Le Bras [ALB21]. More precisely, in the situation of Theorem 1.8, their work exhibits a canonical Hecke eigensheaf $\mathscr{G}_\varphi \in \mathscr{C}_\varphi$ with eigenvalue φ, whose stalk at $B\underline{G_1(\mathbf{Q}_p)}$ can be identified with $\mathcal{F}_{\pi_{\varphi,1}}$. However, the stalks of their sheaf at the other points in $\mathrm{Bun}_G^{\mathrm{ss}}$ are somewhat mysterious. We proceed from the opposite direction, writing down the obvious candidate sheaf $\mathscr{F}_\varphi$ by hand. Forgetting the Weil group action, Theorem 1.6 easily implies that $T_{\mathrm{Std}}\mathscr{F}_\varphi \simeq T_{\mathrm{Std}^\vee}\mathscr{F}_\varphi \simeq \mathscr{F}_\varphi^{\oplus n}$. Moreover, the stalk of $\mathscr{F}_\varphi$ at $B\underline{G_1(\mathbf{Q}_p)}$ identifies with $\mathcal{F}_{\pi_{\varphi,1}}$ by construction, so in particular

$$\mathscr{G}_\varphi | B\underline{G_1(\mathbf{Q}_p)} \simeq \mathscr{F}_\varphi | B\underline{G_1(\mathbf{Q}_p)}.$$

Taken together, this turns out to be just enough information to conclude that $\mathscr{F}_\varphi \simeq \mathscr{G}_\varphi$. Since $\mathscr{G}_\varphi$ is a Hecke eigensheaf, this implies the theorem.

Finally, unwinding the meaning of the Hecke eigensheaf property in Theorem 1.8, we get a considerable generalization of Theorems 1.5 and 1.6. In the following, IH_c^i denotes intersection cohomology with compact support; for the precise definition, we refer the reader to the proof of the subsequent theorem.

Theorem 1.9. *Let (G, μ, b) be any basic local shtuka datum such that G is an inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite extension $L/\mathbf{Q}_p$. Let $\rho \in \mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ be any irreducible representation whose Jacquet-Langlands transfer to $\mathrm{GL}_n(L)$ is supercuspidal. Then as $G(\mathbf{Q}_p) \times W_F$-representations, we have*

$$IH_c^i(G, \mu, b)[\rho] \simeq \begin{cases} \mathrm{JL}(\rho) \boxtimes r_\mu \circ \varphi_\rho|_{W_F} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Here $\mathrm{JL}(\rho)$ is the generalized Jacquet-Langlands transfer of ρ to $G(\mathbf{Q}_p)$.

Note that the definition of the intersection cohomology $IH_c^i(G, \mu, b)[\rho]$ includes a shift and Tate twist, and in particular $IH_c^i(G, \mu, b)[\rho] \cong H_c^{i+d}(G, \mu, b)[\rho](\frac{d}{2})$ when μ is minuscule, so this is indeed a generalization of Theorems 1.5 and 1.6.

We emphasize that the cohomology groups in Theorem 1.9 are completely inaccessible by global methods: for any local shtuka datum (G, μ, b) with μ non-minuscule, the associated local shtuka spaces are entirely unrelated to Shimura varieties! It is therefore rather surprising that Theorem 1.9 can be proved unconditionally. The essential idea here, of bootstrapping from a small amount of globally obtained information using the spectral action constructed in [FS21] combined with the results in [HKW22], seems to be extremely powerful. Indeed, after the first draft of this paper was posted, this technique was subsequently adapted to GSp_4 by Hamann [Ham21] and then to odd unramified unitary groups by Bertoloni–Meli-Hamann-Nguyen [BMHN22]. In both instances, the authors obtain an eigensheaf with all the properties predicted by Fargues, and deduce the strongest possible form of the Kottwitz conjecture for all μ, using Theorem 1.1 as a key input.

1.3. **Perspectives and conjectures.** The global condition in Theorem 1.1 should be irrelevant, but we have not been able to remove it. The following conjecture, formulated in conversations with Scholze, would allow us to circumvent this assumption.

Conjecture 1.10. *For any local Shimura datum (G, μ, b) and any open compact subgroup $K \subset G(\mathbf{Q}_p)$, the local Shimura variety $\mathrm{Sh}(G, \mu, b)_K$ is a Stein space.*

In the setting of Theorem 1.1, the Stein property implies that $R\Gamma_c(G, \mu, b)[\rho]$ is concentrated in degrees $[d, 2d]$ by the Artin vanishing results in [Han20], and then a duality argument (which requires condition 2., cf. Theorem 2.23) finishes the proof. Conjecture 1.10 is known for the Lubin-Tate and Drinfeld cases, for any μ-ordinary local Shimura datum, and for some other cases which can be related to these cases, e.g. in some HN-reducible situations.[2] However, it is entirely open for general local Shimura varieties of unramified EL type. Already the case of $G = \mathrm{GL}_{5/\mathbf{Q}_p}$, $\mu = \mathrm{diag}(z, z, 1, 1, 1)$, and b basic seems very hard.

The following less elegant conjecture would actually have much wider implications.

Conjecture 1.11. *Let $\mathbf{B} = \mathrm{Spa}\, C\left\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty} \right\rangle$ be an affinoid perfectoid ball over some complete algebraically closed field $C/\mathbf{F}_p$, and let $Z \subset \mathbf{B}$ be a Zariski-closed affinoid perfectoid subset. Then $H^i_{\text{ét}}(Z, \mathbf{F}_\ell) = 0$ for all $i > \mathrm{Krull.dim}\,|Z|$ and all $\ell \neq p$.*

Among other things, this would imply the truth of Conjecture 1.7 for all classical groups. This conjecture is widely open, even if Z is cut out by a single equation in a two-dimensional ball.

Conventions. Throughout this paper, we fix a prime p. We write E for a finite extension of $\mathbf{Q}_\ell$ for some prime $\ell \neq p$, and $\mathcal{O}$ for the ring of integers in E. We will assume that E is sufficiently large, where

[2] In particular, the methods in this paper give a new and purely local proof of Theorem 1.1 in the Lubin-Tate and Drinfeld cases, distinct from the arguments in [Har97] and [Mie10]. A variant of this argument was also given by Fargues [Far16a, Theoreme 4.6].

the meaning of "sufficiently large" may change from one line to another; in particular, we do not distinguish between irreducible and absolutely irreducible smooth E-representations of a p-adic reductive group.

Beginning in §2.3, we need to use the étale cohomology of diamonds and v-stacks rather heavily. Here we freely use the formalism developed in [Sch17], in particular the adic formalism defined in [Sch17, §26]. In the notation of loc. cit, we will only apply the adic formalism to the coefficient ring $\Lambda = \mathcal{O}$. We note that the adic formalism of [Sch17, §26] is a "full" six functor formalism, cf. the final paragraph of [Sch17, §26]: in particular, all the usual excision triangles associated with an open-closed decomposition still hold.

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2. Background

2.1. **Isogeny direct summands.** The following notion plays a key technical role in this paper.

Definition 2.1. Let X and Y be objects in an additive category $\mathcal{A}$. We say X is an *isogeny direct summand* of Y if there are maps $i : X \to Y$ and $s : Y \to X$ and a nonzero integer N such that $s \circ i = N \in \mathrm{End}_{\mathcal{A}}(X)$. We say a map $i : X \to Y$ is an *isogeny inclusion* if there exists a map s such that i and s exhibit X as an isogeny direct summand of Y.

In practice, our additive categories will be $\mathbf{Z}_\ell$-linear, so only $N|\ell^\infty$ will be relevant. This notion has several favorable properties.

Proposition 2.2. *1. If X is an isogeny direct summand of Y and Y is an isogeny direct summand of Z, then X is an isogeny direct summand of Z.*

2. If X is an isogeny direct summand of Y and $F : \mathcal{A} \to \mathcal{B}$ is any additive functor of additive categories, then $F(X)$ is an isogeny direct summand of $F(Y)$. If $i : X \to Y$ is an isogeny inclusion, then $F(i)$ is an isogeny inclusion.

3. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{T}$. Suppose that $Nh = 0$ for some nonzero integer N. Then X is an isogeny direct summand of Y and f is an isogeny inclusion. More precisely, there exists a map $\varphi : Y \to X \oplus Z$ such that $\mathrm{Cone}(\varphi)$ is killed by N, and we can choose φ so that $\mathrm{pr}_Z \circ \varphi = g$ and $\mathrm{pr}_X \circ \varphi \circ f = N \in \mathrm{End}_{\mathcal{T}}(X)$.

Conversely, if f is an isogeny inclusion, then $Nh = 0$ for some nonzero integer N.

Proof. 1. and 2. are clear from the definition. For the first part of 3., the axioms of a triangulated category imply that the diagram

$$\begin{tikzcd} Z[-1] \arrow[r, "-h[-1]"] \arrow[d, "\mathrm{id}"] & X \arrow[r, "f"] \arrow[d, "N"] & Y \arrow[r, "g"] & Z \\ Z[-1] \arrow[r, "-Nh[-1]"] & X \end{tikzcd}$$

can be extended to a diagram

$$\begin{tikzcd} Z[-1] \arrow[r, "-h[-1]"] \arrow[d, "\mathrm{id}"] & X \arrow[r, "f"] \arrow[d, "N"] & Y \arrow[r, "g"] \arrow[d, "\varphi"] & {} \\ Z[-1] \arrow[r, "-Nh[-1]"] \arrow[d] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & {} \\ 0 \arrow[r] \arrow[d] & \mathrm{Cone}(X \stackrel{\cdot N}{\to} X) \arrow[r] \arrow[d] & \mathrm{Cone}(X \stackrel{\cdot N}{\to} X) \arrow[r] \arrow[d] & {} \\ {} & {} & {} & \end{tikzcd}$$

where all the rows and columns are distinguished triangles and all squares commute. Since $Nh = 0$ by assumption, we can choose an isomorphism $C \simeq X \oplus Z$ such that the maps $X \to C \to Z$ are the obvious inclusion and projection, cf. [Sta20, Tag 05QT]. The remaining verifications are now easy and are left to the reader.

For the last half of 3., choose some $s : Y \to X$ with $s \circ f = N$. The axioms of a triangulated category imply that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "g"] \arrow[d, "s"] & Z \arrow[r, "h"] & X[1] \\ X \arrow[r, "\cdot N"] & X & & \end{tikzcd}$$

extends to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "g"] \arrow[d, "s"] & Z \arrow[r, "h"] \arrow[d, "t"] & X[1] \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "\cdot N"] & X \arrow[r] & D \arrow[r, "h'"] & X[1] \end{tikzcd}$$

where the rows are exact triangles. Then D is killed by N, so $Nh' = 0$ and then also $Nh = Nh' \circ t = 0$, as desired. □

2.2. **Some representation theory.** In this subsection, we enhance all derived categories to derived ∞-categories.

Let G be a locally pro-p group admitting a countable basis of open compact pro-p subgroups, and fix a prime $\ell \neq p$. Let $E/\mathbf{Q}_\ell$ be some finite extension with ring of integers $\mathcal{O}$ and uniformizer $\varpi \in \mathcal{O}$, so we get the usual Grothendieck abelian categories $\mathrm{Rep}_E(G)$ (resp. $\mathrm{Rep}_{\mathcal{O}}(G)$, resp. $\mathrm{Rep}_{\mathcal{O}/\varpi^n}(G)$) of smooth G-representations on E-vector spaces (resp. on $\mathcal{O}$-modules, resp. on $\mathcal{O}/\varpi^n$-modules). Note that these have natural symmetric monoidal structures. As usual, there are abelian Serre subcategories $\mathrm{Rep}_A(G)_{\mathrm{adm}}$ for $A \in \{E, \mathcal{O}, \mathcal{O}/\varpi^n\}$ consisting of smooth $A[G]$-modules M such that for all open compact subgroups $K \subset G$, M^K is a finitely generated A-module. If $\omega : Z_G \to A^\times$ is a fixed smooth character of the center of G, we write $\mathrm{Rep}_A(G)_\omega$ for the full subcategory of objects with central character ω.

We also need to work with ϖ-adically complete representations. Let $\mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)$ denote the category of ϖ-adically complete $\mathcal{O}[G]$-modules such that $M/\varpi^n M$ is a smooth $\mathcal{O}/\varpi^n[G]$-module for all $n \geq 1$, and let $\mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)_{\mathrm{adm}}$ denote the full subcategory for which the $M/\varpi^n M$'s are admissible smooth.

Proposition 2.3. *The functor* $\mathrm{Rep}_{\mathcal{O}}(G) \to \mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)$ *of naive* ϖ*-adic completion induces an equivalence of categories*

$$\mathrm{Rep}_{\mathcal{O}}(G)_{\mathrm{adm}} \stackrel{\sim}{\to} \mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)_{\mathrm{adm}},$$

with essential inverse given by the functor $M \mapsto \mathrm{colim}_{H\subset G \text{ open}} M^H$ *of smooth vectors. In particular,* $\mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)_{\mathrm{adm}}$ *is an abelian category.*

Proof. Easy and left to the reader. The key point is that for any $M \in \mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)_{\mathrm{adm}}$ and any open compact pro-p subgroup $H \subset G$, the natural map $M^H \to R\lim_n R\Gamma_{\mathrm{cts}}(H, M/\varpi^n M)$ is an isomorphism. For this, observe that $H^i_{\mathrm{cts}}(H, M/\varpi^n M) = 0$ for all $i > 0$ since p is invertible in $\mathcal{O}$, and then $R^1\lim_n (M/\varpi^n M)^H = 0$ since $(M/\varpi^n M)^H$ is a finite abelian group by admissibility. □

We will also need some information about derived categories of representations. For $A \in \{E, \mathcal{O}, \mathcal{O}/\varpi^n\}$, let $D(G, A)$ denote the derived category of $\mathrm{Rep}_A(G)$, and let $D(G, A)_{\mathrm{adm}}$ denote the full subcategory spanned by objects with admissible cohomology modules. On the other hand, fixing a complete algebraically closed field $C/\mathbf{F}_p$ and applying [Sch17, Definitions 14.13 and 26.1] to the classifying stack $B\underline{G} = [\mathrm{Spd}\, C/\underline{G}]$, we get completely natural symmetric monoidal triangulated categories $D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$ and $D_{\text{ét}}(B\underline{G}, \mathcal{O})$, which (morally) consist of complexes of étale sheaves of $\mathcal{O}/\varpi^n$-modules (resp. derived ϖ-complete complexes of étale sheaves of $\mathcal{O}$-modules) on $B\underline{G}$. By Proposition 2.4 below and [Sch17, Proposition 26.2], these categories are independent of the choice of C.

It will be extremely important for us to have some comparison between the geometrically natural categories $D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$ and the more down-to-earth categories $D(G, \mathcal{O}/\varpi^n)$ for $n \leq \infty$. When $n < \infty$ this comparison is as clean as possible.

Proposition 2.4. *There is a natural equivalence of presentably symmetric monoidal stable ∞-categories*

$$D(G, \mathcal{O}/\varpi^n) \cong D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$$
$$A \mapsto \mathcal{F}_A$$

functorial in G and the coefficient ring.

Proof. This is proved in [FS21, Theorem V.1.1]. □

To discuss the case of $\mathcal{O}$-coefficients, we need a "completed" version $\widehat{D}(G, \mathcal{O})$ of $D(G, \mathcal{O})$. It seems somewhat subtle to define a well-behaved category $\widehat{D}(G, \mathcal{O})$ as the actual derived category of some abelian category. Instead, we make the following definition.

Definition 2.5. For any G and $\mathcal{O}$ as above, we set $\widehat{D}(G, \mathcal{O}) = \lim_n D(G, \mathcal{O}/\varpi^n)$, where the limit is computed in the ∞-categorical sense.

Morally (but not literally) this is the subcategory of $D(\mathrm{Mod}_{\mathcal{O}[G]})$ spanned by derived ϖ-complete complexes whose derived mod-ϖ^n reductions are complexes of smooth $\mathcal{O}[G]$-modules.

Proposition 2.6. *1. There is a natural equivalence $\widehat{D}(G, \mathcal{O}) \cong D_{\text{ét}}(B\underline{G}, \mathcal{O})$ functorially in G and $\mathcal{O}$, via a functor denoted $A \mapsto \mathcal{F}_A$.*

2. There is a natural exact symmetric monoidal functor

$$\gamma : D(G, \mathcal{O}) \to \widehat{D}(G, \mathcal{O}).$$
$$A \mapsto (A \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n)_{n \geq 1}.$$

The functor γ commutes with all colimits and therefore admits a right adjoint $\delta : \widehat{D}(G, \mathcal{O}) \to D(G, \mathcal{O})$.

3. For any $A \in D(G, \mathcal{O})_{\mathrm{adm}}$, the unit of the adjunction $A \to \delta\gamma A$ is an isomorphism. In particular, the functor $D(G, \mathcal{O})_{\mathrm{adm}} \to \widehat{D}(G, \mathcal{O})$ is fully faithful.

4. On $\mathcal{O}$-flat representations, the composite functor

$$\mathcal{F}_{-} \circ \gamma : \mathrm{Rep}_{\mathcal{O}}(G) \to D_{\text{ét}}(B\underline{G}, \mathcal{O})$$

factors uniquely over the completion functor $\mathrm{Rep}_{\mathcal{O}}(G) \to \mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)$ via a natural functor

$$\mathcal{F}_{-} : \mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G) \to D_{\text{ét}}(B\underline{G}, \mathcal{O}).$$

The reader should remember that intuitively, γ is completion and δ is decompletion.

Proof. 1. is immediate from the equivalences

$$D_{\text{ét}}(B\underline{G}, \mathcal{O}) \cong \lim_n D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n) \cong \lim_n D(G, \mathcal{O}/\varpi^n)$$

where the first isomorphism follows from [Sch17, Proposition 26.2] and the second isomorphism is immediate from the previous proposition.

For 2., the fact that γ is symmetric monoidal follows as in [Sch17, Remark 26.3]. It is clear from the definition that γ commutes with all direct sums, or equivalently [Lur16, Proposition 1.4.4.1(2)] with all colimits, so the existence of the right adjoint follows from Lurie's ∞-categorical adjoint functor theorem [Lur09, Corollary 5.5.2.9(1)].

The desired functor

$$\mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G) \to D_{\text{ét}}(B\underline{G}, \mathcal{O}) \cong \lim_n D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$$

in 4. sends N to $(\mathcal{F}_{N \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n})_{n\geq 1}$. If $M \in \mathrm{Rep}_{\mathcal{O}}(G)$ is $\mathcal{O}$-flat, the claimed commutativity now follows from the fact that

$$\widehat{M} \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \cong M/\varpi^n M \cong M \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n$$

by $\mathcal{O}$-flatness and [Sta20, Tag 00M9].

For 3., we first note that δ has an explicit description. Precisely, for any open subgroup $H \subset G$ and any $n \leq \infty$, there is a natural functor

$$\begin{aligned} R\Gamma(H, -) : D(G, \mathcal{O}/\varpi^n) &\to D(\mathcal{O}/\varpi^n) \\ A &\mapsto R\mathrm{Hom}(\mathrm{ind}_H^G \mathcal{O}/\varpi^n, A) \end{aligned}$$

of (derived) H-invariants, as well as a completed version

$$\begin{aligned} R\widehat{\Gamma}(H, -) : \widehat{D}(G, \mathcal{O}) &\to D_{\varpi-\mathrm{comp}}(\mathcal{O}) \cong \lim_n D(\mathcal{O}/\varpi^n\mathcal{O}) \to D(\mathcal{O}) \\ (A_n)_{n\geq 1} &\mapsto R\lim_n R\mathrm{Hom}_{D(G,\mathcal{O}/\varpi^n)}(\mathrm{ind}_H^G \mathcal{O}/\varpi^n, A_n). \end{aligned}$$

When H is compact, $R\Gamma(H, -)$ actually coincides with the usual functor of continuous group cohomology on discrete H-modues, hence the notation. Note that $\mathrm{colim}_H R\widehat{\Gamma}(H, B)$ is naturally a complex of smooth $\mathcal{O}[G]$-modules, with G acting by "transport of structure". Then for any $B \in \widehat{D}(G, \mathcal{O})$, $\delta(B)$ is naturally identified with $\mathrm{colim}_H R\widehat{\Gamma}(H, B)$. For this, observe that $\delta B \simeq \mathrm{colim}_H R\Gamma(H, \delta B)$ (this holds with δB replaced by any object of $D(G, \mathcal{O})$). Then observe that $R\Gamma(H, \delta B) = R\widehat{\Gamma}(H, B)$ for any B; this follows by a direct computation from the formula for γ.

Granted this, we compute that

$$\begin{aligned} \delta(\gamma(A)) &\simeq \mathrm{colim}_H R\widehat{\Gamma}(H, \gamma(A)) \\ &\simeq \mathrm{colim}_H \lim_n R\Gamma(H, A \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n) \\ &\simeq \mathrm{colim}_H \lim_n R\Gamma(H, A) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \\ &\simeq \mathrm{colim}_H R\Gamma(H, A) \\ &\simeq A \end{aligned}$$

for any $A \in D(G, \mathcal{O})_{\mathrm{adm}}$. Here the key isomorphism $R\Gamma(H, A) \simeq \lim_n R\Gamma(H, A) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n$ follows from the fact that the cohomology groups of $R\Gamma(H, A)$ are all finitely generated $\mathcal{O}$-modules, which follows from the fact that all $H^i(A)$'s are admissible by assumption. □

Corollary 2.7. *For any $M \in D(G, \mathcal{O})$ and $N \in D(G, \mathcal{O})_{\mathrm{adm}}$, the natural map $R\mathrm{Hom}_{D(G,\mathcal{O})}(M, N) \to R\mathrm{Hom}_{\widehat{D}(G,\mathcal{O})}(\gamma M, \gamma N)$ is an isomorphism.*

Proof. By part 3. of the previous proposition, we have $R\mathrm{Hom}_{D(G,\mathcal{O})}(M, N) \cong R\mathrm{Hom}_{D(G,\mathcal{O})}(M, \delta\gamma N)$, so the result follows by adjunction. □

Lemma 2.8. *Let M be an admissible smooth ℓ-torsion-free $\mathcal{O}[G]$-module. Let $\pi \in \mathrm{Rep}_E(G)$ be a finite-length admissible smooth representation of G. Suppose moreover that π is a direct summand of $M[\frac{1}{\ell}]$. Then π admits a G-stable $\mathcal{O}$-lattice π°, and any such lattice is an isogeny direct summand of M.*

In what follows, when we say "$\mathcal{O}$-lattice", we mean "G-stable $\mathcal{O}$-lattice".

Proof. Since π is of finite length, it is finitely generated, so some results of Vignéras [Vig04] imply that any $\mathcal{O}$-lattice in π is finitely generated as an $\mathcal{O}[G]$-module, and any two $\mathcal{O}$-lattices in π are commensurable. Moreover, an $\mathcal{O}[G]$-submodule of π is an $\mathcal{O}$-lattice if and only if it is a free $\mathcal{O}$-module and it contains an E-basis of π. These results easily imply that if *some* $\mathcal{O}$-lattice $\pi^\circ \subset \pi$ is an isogeny direct summand of M, then *all* $\mathcal{O}$-lattices are isogeny direct summands of M. Thus, we only need to exhibit a single such π°.

By assumption, we can choose morphisms $i : \pi \to M[\frac{1}{\ell}]$ and $s : M[\frac{1}{\ell}] \to \pi$ with $s \circ i = \mathrm{id}$. We claim that $\pi^\circ = s(M)$ is an $\mathcal{O}$-lattice in π. Clearly π° contains an E-basis of π, so it suffices to check that π° is a free $\mathcal{O}$-module. Choose a countable cofinal set of open pro-p subgroups

$$\cdots \lhd K_n \lhd \cdots \lhd K_3 \lhd K_2 \lhd K_1 \subset G.$$

Then each $s(M^{K_n}) = s(M)^{K_n}$ is a finite free $\mathcal{O}$-module, and the inclusions $s(M)^{K_n} \to s(M)^{K_{n+1}}$ are split as $\mathcal{O}$-modules by the obvious averaging maps, using that $[K_{n+1} : K_n] \in \mathcal{O}^\times$. Therefore $\pi^\circ = \mathrm{colim}_n s(M)^{K_n}$ is a free $\mathcal{O}$-module.

Since π° is a finitely generated $\mathcal{O}[G]$-submodule, we can choose some large n such that $\ell^n i(\pi^\circ) \subseteq M$. Setting $i' = \ell^n i$, we now have a pair of maps $i' : \pi^\circ \to M$ and $s : M \to \pi^\circ$ with $s \circ i = \ell^n$, so these maps exhibit π° as an isogeny direct summand of M. □

As a consequence, we can globalize lattices in square-integrable representations. For simplicity we only discuss the supercuspidal case.

Proposition 2.9. *Let $\mathbf{H}/\mathbf{Q}$ be a connected reductive group such that $\mathbf{H}(\mathbf{R})$ is compact modulo center. Fix an isomorphism $\iota : \overline{E} \to \mathbf{C}$, and let $\pi \in \mathrm{Rep}_E(\mathbf{H}(\mathbf{Q}_p))$ be any irreducible supercuspidal representation. Then some unramified twist of π admits a $\mathbf{H}(\mathbf{Q}_p)$-stable $\mathcal{O}$-lattice, and this lattice occurs as an isogeny direct summand in a space of algebraic automorphic forms for $\mathbf{H}$.*

More precisely, we can choose the data of

- *An open compact subgroup $K^p \subset \mathbf{H}(\mathbf{A}^{f,p})$,*
- *An irreducible algebraic $\mathbf{H}$-representation $\mathcal{L}_\xi$ of some regular highest weight ξ,*
- *A K^p-stable $\mathcal{O}$-lattice $\mathcal{L}_{\xi,\mathcal{O}}$ in the E-linear realization $\mathcal{L}_{\xi,E}$ of $\mathcal{L}_\xi$ (where K^p acts via the composition $K^p \to \mathbf{H}(\mathbf{A}^f) \to \mathbf{H}(\mathbf{Q}_\ell)$), and*
- *A $\mathbf{H}(\mathbf{Q}_p)$-stable $\mathcal{O}$-lattice τ° in an unramified twist $\tau = \pi \otimes \eta$ of π,*

such that τ° is an isogeny direct summand of

$$\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \overset{\mathrm{def}}{=} \operatorname*{colim}_{K_p \to \{1\}} \mathcal{A}(\mathbf{H}(\mathbf{Q})\backslash \mathbf{H}(\mathbf{A}^f)/K^p K_p, \mathcal{L}_{\xi,\mathcal{O}})$$

in the category of admissible smooth $\mathcal{O}[\mathbf{H}(\mathbf{Q}_p)]$-modules.

Proof. By construction, $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbf{C}$ is a space of algebraic automorphic forms for $\mathbf{H}$ with infinite level at p. Since $\iota\pi$ is essentially square-integrable, a standard argument with the simple trace formula shows that we may choose ξ and K^p such that some unramified twist $\iota\pi \otimes \eta'$ occurs as a direct summand of $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbf{C}$ (cf. [Shi12a, Proposition 5.3 and Example 5.6] for a very general result along these lines). Replacing E by a sufficiently large finite extension, we can assume that $\eta = \iota^{-1}\eta'$ is valued in E, so then $\tau = \pi \otimes \eta$ occurs as a direct summand of $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}})[\frac{1}{\ell}]$. Since $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}})$ is an admissible smooth ℓ-torsion-free $\mathcal{O}[\mathbf{H}(\mathbf{Q}_p)]$-module, we now conclude by Lemma 2.8. □

2.3. **Some p-adic geometry.** In this subsection, we give some recollections on p-adic geometry. Fix a complete algebraically closed extension $C/\mathbf{Q}_p$. All rigid spaces will live over $\mathrm{Spa}\, C$, and all diamonds, small v-stacks, etc. will live over $\mathrm{Spd}\, C$. If G is a locally profinite group, we write $B\underline{G} = [\mathrm{Spd}\, C/\underline{G}]$.

Fix a reductive group $G/\mathbf{Q}_p$, and let Bun_G be the stack of G-bundles on the Fargues-Fontaine curve. In our convention, this is a category fibered in groupoids over $\mathrm{Perf}_{/\mathrm{Spd}\,C} \cong \mathrm{Perfd}_{/\mathrm{Spa}\,C}$. Recall that Bun_G is a small v-stack, and by a theorem of Fargues [Far15] there is a natural continuous bijection

$$|\mathrm{Bun}_G| \xrightarrow{\sim} B(G)$$
$$x_b \leftarrow\!\shortmid b$$

where $B(G)$ is topologized by the partial order topology on Newton points. Moreover, for any basic b there is a natural open immersion $j : B\underline{G_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$.

Now fix a geometric conjugacy class of minuscule cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$, and let $\mathscr{F}\ell_{G,\mu}$ be the associated rigid analytic flag variety. By [CS17, §3.4-3.5], we can also identify $\mathscr{F}\ell_{G,\mu}$ with the functor $\mathrm{Perfd}_{/\mathrm{Spa}\,C} \to \mathrm{Sets}$ sending any S to the set of isomorphism classes of μ-bounded meromorphic modifications

$$(\mathcal{E}, i : \mathcal{E}_{\mathbf{1}}|_{\mathcal{X}_{S^\flat} \smallsetminus S} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{X}_{S^\flat} \smallsetminus S})$$

of G-bundles on the relative Fargues-Fontaine curve $\mathcal{X}_{S^\flat}$. In particular, there is a canonical map $\tilde{q} : \mathscr{F}\ell_{G,\mu} \to \mathrm{Bun}_G$ sending the data of such a modification to the class of the bundle $\mathcal{E}$.

Proposition 2.10. *1. The map $\tilde{q}$ factors as $\mathscr{F}\ell_{G,\mu} \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \xrightarrow{q} \mathrm{Bun}_G$, and the map q is ℓ-cohomologically smooth for all $\ell \neq p$.*

2. For any open compact subgroup $K \subset G(\mathbf{Q}_p)$, the map

$$q_K : [\mathscr{F}\ell_{G,\mu}/\underline{K}] \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \xrightarrow{q} \mathrm{Bun}_G$$

is ℓ-cohomologically smooth for all $\ell \neq p$.

Proof. For 1., recall that the $G(\mathbf{Q}_p)$-action on $\mathscr{F}\ell_{G,\mu}$ changes a modification $(\mathcal{E}, i)$ by precomposing i with the action of $G(\mathbf{Q}_p) \cong \mathrm{Aut}(\mathcal{E}_{\mathbf{1}})$. Since the map $\tilde{q}$ discards the data of i, the claimed factorization is clear. For the smoothness claim, one checks that for any perfectoid space S/C with a map $S \to \mathrm{Bun}_G$, the fiber product

$$[\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \times_{\mathrm{Bun}_G} S$$

is étale-locally on S isomorphic to an open subspace of $\mathscr{F}\ell_{G,\mu^{-1}} \times_{\mathrm{Spd}\,C} S$, which is then ℓ-cohomologically smooth over S. Part 2. is now clear, since $[\mathscr{F}\ell_{G,\mu}/\underline{K}] \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}]$ is étale, hence ℓ-cohomologically smooth, and compositions of cohomologically smooth maps are cohomologically smooth. □

Recall from [CS17] that $\mathscr{F}\ell_{G,\mu}$ admits a $G(\mathbf{Q}_p)$-invariant Newton stratification into locally spatial locally closed subdiamonds $\mathscr{F}\ell^b_{G,\mu} \subset \mathscr{F}\ell_{G,\mu}$ indexed by $b \in B(G,\mu)$. By definition, these are the subdiamonds corresponding to the subspaces $|\tilde{q}|^{-1}(x_b) \subset |\mathscr{F}\ell_{G,\mu}|$ for any individual $b \in B(G) = |\mathrm{Bun}_G|$.

The following observation settles a question left open in [CS17], although we won't need it in this paper.

Proposition 2.11. *The stratification $|\mathscr{F}\ell_{G,\mu}| = \coprod_{b \in B(G,\mu)} |\mathscr{F}\ell^b_{G,\mu}|$ is a true stratification, in the sense that the closure of any stratum is a union of strata. Moreover, if $G = \mathrm{GL}_n$ then $\overline{|\mathscr{F}\ell^b_{G,\mu}|} = \coprod_{b' \geq b} |\mathscr{F}\ell^{b'}_{G,\mu}|$.*

Proof. As above, the map $\tilde{q}$ factors as a $G(\mathbf{Q}_p)$-torsor followed by a cohomologically smooth map, so $|\tilde{q}|$ is an open map. The first claim now follows from the fact that $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ for any open continuous map of topological spaces, together with the fact that the decomposition $|\mathrm{Bun}_G| = \coprod_{b \in B(G)} \{x_b\}$ is obviously a true stratification. The second claim then follows from [Han17]. □

We now turn to local Shimura varieties. Choose some element $b \in B(G,\mu)$, so (G,μ,b) is a local Shimura datum.

Definition 2.12. The local Shimura variety $\mathrm{Sh}(G,\mu,b)_\infty$ is the functor on perfectoid spaces over $\mathrm{Spa}\,C$ sending any S to the set of μ-bounded meromorphic modifications

$$\mathcal{E}_{\mathbf{1}}|_{\mathcal{X}_{S^\flat} \smallsetminus S} \xrightarrow{\sim} \mathcal{E}_b|_{\mathcal{X}_{S^\flat} \smallsetminus S}.$$

For $K \subset G(\mathbf{Q}_p)$ an open compact subgroup, $\mathrm{Sh}(G,\mu,b)_K$ is the unique rigid analytic space such that $\mathrm{Sh}(G,\mu,b)_K^\diamond = \mathrm{Sh}(G,\mu,b)_\infty/\underline{K}$.

It turns out that for any given representation $\rho^\circ \in \mathrm{Rep}_{\mathcal{O}}(G_b(\mathbf{Q}_p))_{\mathrm{adm}}$, there are several natural ways to define "the ρ°-part of the cohomology of the tower $\{\mathrm{Sh}(G,\mu,b)_K\}_K$". Fortunately these recipes all agree, up to the essentially harmless difference between complete admissible representations and smooth admissible representations. Before explaining this, we need to define the cohomology of the individual spaces $\mathrm{Sh}(G,\mu,b)_K$. Here there are already two natural candidate definitions.

Definition 2.13. Let X be any compactifiable locally spatial diamond of finite dim.trg over C, with structure morphism $f : X \to \mathrm{Spd}\, C$. Then for any $n \leq \infty$, we define $R\Gamma_c(X, \mathcal{O}/\varpi^n) = Rf_!(\mathcal{O}/\varpi^n)$, where $Rf_!$ is defined as in [Sch17, §22 and §26]. The assignment $X \mapsto R\Gamma_c(X,\mathcal{O})$ is covariant for open immersions, so we can also define

$$R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O}) = \mathrm{colim}_{U\subset X} R\Gamma_c(U,\mathcal{O})$$

where the colimit runs over all quasicompact open subspaces $U \subset X$.[3]

Note that we could obviously also define $R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O}/\varpi^n)$ for finite n, but this coincides with $R\Gamma_c(X,\mathcal{O}/\varpi^n)$.

Proposition 2.14. *1. In the notation of the previous definition, there is a natural map*

$$R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O}) \to R\Gamma_c(X,\mathcal{O})$$

identifying the target with the derived ℓ-completion of the source.

2. If X is ℓ-cohomologically smooth and equipped with a $\underline{G}$-action for some locally pro-p group G, then $R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O})$ is naturally a complex of smooth $\mathcal{O}[G]$-modules, and $R\Gamma_c(X,\mathcal{O}) = \gamma R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O})$ where γ is the completion functor defined in Proposition 2.6.3.

Note that in the setup of part 2., one gets a map $R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O}) \to \delta R\Gamma_c(X,\mathcal{O})$ by adjunction. It is plausible that this map is often an isomorphism, but we didn't try to check this.

Proof. 1. is immediate form the observation that

$$R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \to R\Gamma_c(X,\mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \cong R\Gamma_c(X,\mathcal{O}/\varpi^n)$$

is an isomorphism. For the first half of 2., note that any qc open $U \subset X$ is stable under an open pro-p subgroup $K \subset G$. Then $f : [U/\underline{K}] \to B\underline{K}$ is qcqs and ℓ-cohomologically smooth, so $Rf_!$ preserves compact objects as its right adjoint $Rf^!$ commutes with all direct sums. Thus $Rf_!\mathcal{O} = R\Gamma_c(U,\mathcal{O}) \in D_{\text{ét}}(B\underline{K},\mathcal{O})$ is a compact object, and in particular it is a bounded complex of finitely generated smooth $\mathcal{O}[K]$-modules. Passing to the colimit, we deduce that $R\Gamma_{c,\mathrm{sm}}(X,\mathcal{O})$ is naturally a complex of smooth $\mathcal{O}[G]$-modules. The second half of 2. is immediate from the definition of γ. □

Definition 2.15. For (G,μ,b) a local Shimura datum and $K \subset G(\mathbf{Q}_p)$ an open compact subgroup, we define

$$R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E) := R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G,\mu,b)_K,\mathcal{O}) \otimes_{\mathcal{O}} E$$

and

$$R\Gamma_c(\mathrm{Sh}(G,\mu,b), E) = \operatorname*{colim}_{K\to\{1\}} R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E).$$

By the previous proposition, $R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E)$ is a complex of smooth $G_b(\mathbf{Q}_p)$-representations, and $R\Gamma_c(\mathrm{Sh}(G,\mu,b), E)$ is a complex of smooth $G(\mathbf{Q}_p) \times G_b(\mathbf{Q}_p)$-representations.

We now restrict our attention to the case where $b \in B(G,\mu)$ is the unique basic element.

Proposition 2.16. *Let $b \in B(G,\mu)$ be the unique basic element. For any open compact subgroup $K \subset G(\mathbf{Q}_p)$ there is a natural Cartesian diagram*

$$\begin{tikzcd}
\mathrm{Sh}(G,\mu,b)_K \arrow[r] \arrow[d] & {[\mathscr{F}\ell^b_{G,\mu}/\underline{K}]} \arrow[d, "q^b_K"] \\
\mathrm{Spd}\, C \arrow[r] & B\underline{G_b(\mathbf{Q}_p)}
\end{tikzcd}$$

[3] The colimit here is computed in the derived ∞-category $D(\mathcal{O})$.

compatibly with varying K. Here the right-hand vertical map is the pullback of $q_K : [\mathscr{F}\ell_{G,\mu}/\underline{K}] \to \mathrm{Bun}_G$ along the open immersion $B\underline{G_b(\mathbf{Q}_p)} \subset \mathrm{Bun}_G$.

Proof. Clear from the definitions. □

Now, fix an admissible representation $\rho^\circ \in \mathrm{Rep}_{\mathcal{O}}(G_b(\mathbf{Q}_p))_{\mathrm{adm}}$.

Proposition 2.17. *Notation as above, consider the following complexes of $\mathcal{O}[G(\mathbf{Q}_p)]$-modules.*

1. $M_1 = \mathrm{colim}_K R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G,\mu,b)_K, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(G_b(\mathbf{Q}_p))} \rho^\circ$.
2. $M_2 = R\Gamma_c(\mathscr{F}\ell^b_{G,\mu}, \tilde{q}^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$, *where* $\tilde{q}^b : \mathscr{F}\ell^b_{G,\mu} \to B\underline{G_b(\mathbf{Q}_p)}$ *is the obvious pullback of* $\tilde{q}$.
3. $M_3 = \mathrm{colim}_K R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$, *where* $\tilde{j} : [\mathscr{F}\ell^b_{G,\mu}/\underline{K}] \to [\mathscr{F}\ell_{G,\mu}/\underline{K}]$ *is the natural open immersion.*

Then M_1 and M_3 are bounded complexes of smooth $\mathcal{O}[G(\mathbf{Q}_p)]$-modules with admissible cohomology, and M_2 is a bounded derived ℓ-complete complex with complete admissible cohomology. Moreover, there are natural maps $M_1 \to M_2 \leftarrow M_3$ which become isomorphisms after derived ℓ-completion.

Applying Proposition 2.6.3 twice, we deduce that there is a natural quasi-isomorphism $M_1 \cong M_3$ in $D(G(\mathbf{Q}_p), \mathcal{O})$. Note that $M_1[\frac{1}{\ell}]$ coincides with $R\Gamma_c(G,\mu,b)[\rho]$ as defined in the introduction. On the other hand, $M_3[\frac{1}{\ell}]$ is the object which will emerge naturally in our proof of Theorem 1.1.

Proof. We first construct the maps. To build the map $M_1 \to M_2$, note that there is an obvious map

$$\mathrm{colim}_K R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G,\mu,b)_K, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(G_b(\mathbf{Q}_p))} \rho^\circ \to R\Gamma_c(\mathrm{Sh}(G,\mu,b)_\infty, \mathcal{O}) \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{H}}_{\mathcal{O}}(G_b(\mathbf{Q}_p))} \widehat{\rho^\circ} \overset{\mathrm{def}}{=} N.$$

Indeed, the right-hand side is the derived ℓ-completion of the left-hand side. On the other hand, the local Hodge-Tate period map $\mathrm{Sh}(G,\mu,b)_\infty \to \mathscr{F}\ell^b_{G,\mu}$ is a pro-étale $G_b(\mathbf{Q}_p)$-torsor, so by Lemma 2.18 below there is a natural isomorphism $N \cong M_2$.

To build the map $M_3 \to M_2$, note that by general nonsense, the natural map

$$\mathrm{colim}_K R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*}\mathcal{F}_{\widehat{\rho^\circ}}) \to R\Gamma(\mathscr{F}\ell_{G,\mu}, \tilde{j}_! \tilde{q}^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$$

identifies the target with the derived ℓ-completion of the source.[4] On the other hand,

$$R\Gamma_c(\mathscr{F}\ell^b_{G,\mu}, q^{b*}\mathcal{F}_{\widehat{\rho^\circ}}) \cong R\Gamma(\mathscr{F}\ell_{G,\mu}, \tilde{j}_! q^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$$

since $\mathscr{F}\ell_{G,\mu}$ is proper.

We've already shown that the maps become isomorphisms after derived ℓ-completion. Thus, by two applications of Proposition 2.6.3, it remains to see that M_3 is admissible, i.e. that for any given K

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$$

is a perfect complex of $\mathcal{O}$-modules. This follows from the admissibility of ρ° and the fundamental finiteness theorems proved in [FS21, §IX.3]. □

In the previous proof, we used the following lemma.

Lemma 2.18. *Let X/C be a compactifiable locally spatial diamond of finite dim.trg, and let $\tilde{X} \to X$ be a G-torsor for some locally pro-p-group G with $p \neq \ell$, corresponding to a map $q : X \to B\underline{G}$. Let $A \in \widehat{D}(G,\mathcal{O})$ be any object, with $\mathcal{F}_A \in D_{\acute{e}t}(B\underline{G},\mathcal{O})$ the associated sheaf. Then there is a natural isomorphism*

$$R\Gamma_c(X, q^*\mathcal{F}_A) \cong R\Gamma_c(\tilde{X}, \mathcal{O}) \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{H}}_{\mathcal{O}}(G)} A.$$

Proof. This follows from an easy ℓ-complete variant of the analysis in [Man04, §5] and [HK19, Appendix B]. □

Finally, we need to discuss inert representations.

[4] Let X_i be a cofiltered system of qcqs small v-stacks with 0-truncated qcqs transition maps indexed by a category I with an initial object 0. Let $X = \lim X_i$ be the limiting small v-stack. Let $\mathcal{F}_0 \in D_{\acute{e}t}(X_0,\mathcal{O})$ be some object, with pullbacks $\mathcal{F}_i \in D_{\acute{e}t}(X_i,\mathcal{O})$ and $\mathcal{F} \in D_{\acute{e}t}(X,\mathcal{O})$. Then there is a natural map $\mathrm{colim}_i R\Gamma(X_i,\mathcal{F}_i) \to R\Gamma(X,\mathcal{F})$ identifying the target with the derived ℓ-completion of the source.

Definition 2.19. Let $b \in B(G)_{\mathrm{bas}}$ be any basic element, with $j : \underline{BG_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ the associated open immersion. A smooth admissible representation $\pi^\circ \in \mathrm{Rep}_{\mathcal{O}}(G_b(\mathbf{Q}_p))_{\mathrm{adm}}$ is *inert* if for every quasicompact open substack $W \subset \mathrm{Bun}_G$,

$$\mathrm{Cone}(j_! \mathcal{F}_{\widehat{\pi^\circ}} \to Rj_* \mathcal{F}_{\widehat{\pi^\circ}})|_W \in D_{\text{ét}}(W, \mathcal{O})$$

is killed by ℓ^n for some $n \gg_W 0$.

Proposition 2.20. *Suppose that $\pi \in \mathrm{Rep}_E(J_b(\mathbf{Q}_p))_{\mathrm{adm}}$ is finitely generated and admissible. If some $\mathcal{O}$-lattice $\pi^\circ \subset \pi$ is inert, then every $\mathcal{O}$-lattice in π is inert.*

Proof. This follows easily from the commensurability of any two $\mathcal{O}$-lattices in π [Vig04]. □

Definition 2.21. Let $\pi \in \mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ be any irreducible representation. We say π is inert if some unramified twist of π contains an inert $\mathcal{O}$-lattice.

We will need the following crucial result.

Theorem 2.22. *Suppose that $\rho \in \mathrm{Irr}_E(G_b(\mathbf{Q}_p))$ is an irreducible representation admitting an $\mathcal{O}$-lattice, and that the L-parameter*

$$\varphi_\rho : W_{\mathbf{Q}_p} \to {}^L G_b(\overline{\mathbf{Q}_\ell}) \cong {}^L G(\overline{\mathbf{Q}_\ell})$$

associated with ρ by [FS21] is supercuspidal. Then ρ is supercuspidal, and every $\mathcal{O}$-lattice $\rho^\circ \subset \rho$ is inert.

The proof of this theorem relies on the full power of the machinery developed in [FS21]. In the following argument, we will freely use various notation and results from [FS21, Ch. VIII-IX].

Proof. Supercuspidality of ρ follows from [FS21, Corollary IX.7.3]. It remains to prove inertness. Via the isomorphism $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_b}$, we can assume that $b = 1$ and $G_b = G$. Fix an $\mathcal{O}$-lattice $\rho^\circ \subset \rho$, and let $\mathcal{F}_{\widehat{\rho^\circ}}$ be the associated sheaf on $\underline{BG(\mathbf{Q}_p)}$. It suffices to prove that for every non-basic b with associated stratum $i_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$, the object $i_b^* Rj_* \mathcal{F}_{\widehat{\rho^\circ}} \in \widehat{D}(G_b(\mathbf{Q}_p), \mathcal{O})$ is killed by ℓ^n for some n. In fact, the following argument will show that n can be chosen independently of b.

For Λ any $\mathcal{O}$-algebra, let $\mathcal{Z}^{\mathrm{spec}}(G)_\Lambda := \mathcal{O}(Z^1(W_{\mathbf{Q}_p}, \widehat{G})_\Lambda)^{\widehat{G}}$ be the spectral Bernstein center with coefficients in Λ. As in [FS21, Definition IX.7.1], there is a canonical map from $\mathcal{Z}^{\mathrm{spec}}(G)_{\mathcal{O}}$ to the Bernstein center $\mathcal{Z}(D(G(\mathbf{Q}_p), \mathcal{O}))$, which induces a composite map to the Bernstein center of $\widehat{D}(G(\mathbf{Q}_p), \mathcal{O})$. Moreover, by [FS21, Theorem IX.7.2], the induced diagram

$$\begin{tikzcd}
\mathcal{Z}^{\mathrm{spec}}(G)_{\mathcal{O}} \arrow[r, "\Psi_G"] \arrow[d, "r"] & \mathrm{End}(\mathcal{F}_{\widehat{\rho^\circ}}) \cong \mathrm{End}(\rho^\circ) \arrow[d, "s"] \\
\mathcal{Z}^{\mathrm{spec}}(J_b)_{\mathcal{O}} \arrow[r, "\Psi_{G_b}"] & \mathrm{End}(i_b^* Rj_* \mathcal{F}_{\widehat{\rho^\circ}})
\end{tikzcd}$$

commutes. Here r is induced by the natural map $\widehat{G_b} \to \widehat{G}$ realizing $\widehat{G_b}$ as a Levi subgroup of $\widehat{G}$.

Now, let $e_\rho \in \mathcal{Z}^{\mathrm{spec}}(G)_E$ be the idempotent cutting out the connected component C_ρ of $Z^1(W_{\mathbf{Q}_p}, \widehat{G})_E$ containing the parameter φ_ρ. Let X be the connected component of $Z^1(W_{\mathbf{Q}_p}, \widehat{G})_{\mathcal{O}}$ containing the image of C_ρ, so $\mathcal{O}(X)$ resp. $\mathcal{O}(X)[1/\ell]$ is naturally a summand of $\mathcal{Z}^{\mathrm{spec}}(G)_{\mathcal{O}}$ resp. $\mathcal{Z}^{\mathrm{spec}}(G)_E$, and $e_\rho \in \mathcal{O}(X)[1/\ell]$. Quite generally, if X is a reduced finite type $\mathcal{O}$-scheme and $C \subset X_E$ is a connected component with associated idempotent $e \in \mathcal{O}(X)[1/\ell]$, there is some integer n such that $\ell^n e \in \mathcal{O}(X)$. Applying this in the situation at hand, we can choose some n such that $\ell^n e_\rho \in \mathcal{Z}^{\mathrm{spec}}(G)_{\mathcal{O}}$. Then $\Psi_G(\ell^n e_\rho) = \ell^n$ as endomorphisms of ρ°: this can be checked after inverting ℓ, where it reduces to the fact that e_ρ acts as the identity on ρ under the natural map $\mathcal{Z}^{\mathrm{spec}}(G)_E \to \mathcal{Z}(D(G(\mathbf{Q}_p), E))$. On the other hand, $r(\ell^n e_\rho) = 0$, since the component of $Z^1(W_{\mathbf{Q}_p}, \widehat{G})_E // \widehat{G}$ containing the closed point corresponding to φ_ρ is disjoint from the image of

$$Z^1(W_{\mathbf{Q}_p}, \widehat{M})_E // \widehat{M} \to Z^1(W_{\mathbf{Q}_p}, \widehat{G})_E // \widehat{G}$$

for any proper Levi $\widehat{M} \subset \widehat{G}$. Going around the diagram, we then compute that

$$\begin{aligned} \ell^n &= s(\ell^n) \\ &= (s \circ \Psi_G)(\ell^n e_\rho) \\ &= (\Psi_{J_b} \circ r)(\ell^n e_\rho) \\ &= 0 \end{aligned}$$

in $\mathrm{End}(i_b^* Rj_* \mathcal{F}_{\widehat{\rho^\circ}})$, as desired. □

When ρ is inert, $R\Gamma_c(G,\mu,b)[\rho]$ satisfies a clean duality principle.

Theorem 2.23. *Fix a basic local Shimura datum (G,μ,b), and let ρ be an inert representation of $G_b(\mathbf{Q}_p)$. Writing $(-)^*$ for the contragredient, there is a natural isomorphism*

$$R\Gamma_c(G,\mu,b)[\rho]^* \cong \left(R\Gamma_c(G,\mu,b)[\rho^*]\right)[2d](d)$$

of W_F-equivariant objects in $D(G(\mathbf{Q}_p),E)$. In particular, there is a natural W_F-equivariant isomorphism

$$H^i_c(G,\mu,b)[\rho]^* \cong H^{2d-i}_c(G,\mu,b)[\rho^*](d)$$

of smooth $G(\mathbf{Q}_p)$-representations.

Proof. Replacing ρ by a twist, we can assume that ρ contains an inert $\mathcal{O}$-lattice ρ°. By Proposition 2.17, it's enough to produce a natural map

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^{\circ *}}})[2d](d) \to R\mathrm{Hom}(R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^\circ}}), \mathcal{O})$$

functorially in open pro-p subgroups $K \subset G(\mathbf{Q}_p)$ whose cone is killed by ℓ^n for some $n \gg 0$ independent of K. For this, look at the diagram

$$\begin{tikzcd} & {[\mathscr{F}\ell_{G,\mu}/\underline{K}]} \arrow[dl, "p_K"'] \arrow[dr, "q_K"] & \\ B\underline{K} & & \mathrm{Bun}_G \end{tikzcd}$$

where q_K is smooth and p_K is proper and smooth. An exercise with the six functors shows that

$$R\mathrm{Hom}(R\Gamma(B\underline{K}, Rp_{K*} q_K^* \mathcal{G}), \mathcal{O}) \cong R\Gamma(B\underline{K}, (Rp_{K*} q_K^* \mathbf{D}\mathcal{G})[2d](d))$$

for any $\mathcal{G} \in D_{\text{ét}}(\mathrm{Bun}_G, \mathcal{O})$, where $\mathbf{D}\mathcal{G} = R\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{O})$ denotes the Verdier dual.[5] Applying this with $\mathcal{G} = j_! \mathcal{F}_{\widehat{\rho^\circ}}$, in which case $\mathbf{D}\mathcal{G} = Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}}$ we deduce that

$$R\mathrm{Hom}(R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^\circ}}), \mathcal{O}) \cong R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], q_K^* Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}})[2d](d).$$

But inert representations are preserved under duality, and the image of q_K is quasicompact and independent of K, so the cone of the natural map

$$\tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^{\circ *}}} \cong q_K^* j_! \mathcal{F}_{\widehat{\rho^{\circ *}}} \to q_K^* Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}}$$

is killed by some ℓ^n with n independent of K. Applying $R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], -)$ gives the desired map. □

Corollary 2.24. *If $\rho \in \mathrm{Rep}_E(G_b(\mathbf{Q}_p))$ is an inert supercuspidal representation, there is a natural isomorphism*

$$R\Gamma_c(G,\mu,b)[\rho] \cong \operatorname*{colim}_K R\mathrm{Hom}_{D(G_b(\mathbf{Q}_p),E)}\left(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E)[2d](d), \rho\right).$$

In particular, when ρ is inert, $R\Gamma_c(G,\mu,b)[\rho]$ as defined in this paper coincides with $R\Gamma(G,\mu,b)[\rho][-d](\frac{-d}{2})$, where $R\Gamma(G,\mu,b)[\rho]$ is defined as in [HKW22].

[5] Strictly speaking, we are also using the duality $R\Gamma(B\underline{K},-)^\vee \cong R\Gamma(B\underline{K},(-)^\vee)$, which is an easy exercise using the (canonical) $\mathcal{O}$-valued Haar measure on K in this situation.

Proof. It is clear from the definitions that

$$R\Gamma_c(G,\mu,b)[\rho]^* \cong \mathrm{colim}_K R\mathrm{Hom}_{D(G_b(\mathbf{Q}_p),E)}(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E),\rho^*)$$

for any ρ. When ρ is inert, the left-hand side is identified with $(R\Gamma_c(G,\mu,b)[\rho^*])[2d](d)$ by the previous theorem, so the result follows by moving the shifts and twists appropriately. $\square$

3. Proofs

3.1. **Basic uniformization.** Let $(\mathbf{G},X)$ be a Shimura datum, with conjugacy class of inverse Hodge cocharacters $\mu : \mathbf{G}_{m,\mathbf{C}} \to \mathbf{G}_{\mathbf{C}}$. Fix a prime p, and set $G = \mathbf{G} \otimes_{\mathbf{Q}} \mathbf{Q}_p$. Via our fixed isomorphisms $\mathbf{C} \simeq \overline{\mathbf{Q}_p}$, we can and do regard μ as a conjugacy class of cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$. In particular, we may speak of the Kottwitz set $B(G,\mu)$.

Proposition 3.1. *Let $(\mathbf{G},X)$ be a Shimura datum, and let p be any prime. Then with notation as above, $\mathbf{G}$ admits a canonical $\mathbf{Q}$-inner form $\mathbf{G}'$ such that*

i. $\mathbf{G}'_{\mathbf{A}^{\{\infty p\}}} \simeq \mathbf{G}_{\mathbf{A}^{\{\infty p\}}}$ *as algebraic groups over* $\mathbf{A}^{\{p\infty\}}$,

ii. $\mathbf{G}'_{\mathbf{Q}_p} \simeq G_b$, *where $G_b/\mathbf{Q}_p$ is the inner form of G associated with the unique basic element $b \in B(G,\mu)$, and*

iii. $\mathbf{G}'(\mathbf{R})$ *is compact modulo center.*

We emphasize that $\mathbf{G}'$ depends on the pair $(\mathbf{G},X)$ and not just on $\mathbf{G}$. When $(\mathbf{G},X)$ is a Hodge type Shimura datum with good reduction at p, this was proved by Xiao-Zhu [XZ17, Corollary 7.2.15]. The following general argument was suggested by Zhiyu Zhang.

Proof. By the Hasse principle for adjoint groups [PR94, Theorem 6.22] and the analysis in [Kot86, §1-2], there is a natural exact sequence

$$1 \to H^1(\mathbf{Q},\mathbf{G}^{\mathrm{ad}}) \to \oplus_{v\leq\infty} H^1(\mathbf{Q}_v,\mathbf{G}^{\mathrm{ad}}) \xrightarrow{\alpha} \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}}})^D$$

of pointed sets, where $(-)^D$ denotes the Pontryagin dual. Here the map α is obtained as the direct sum of the canonical maps

$$\alpha_{\mathbf{G}^{\mathrm{ad}},v} : H^1(\mathbf{Q}_v,\mathbf{G}^{\mathrm{ad}}) \to \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}_v}})^D$$

defined by Kottwitz, composed with the map

$$\oplus_v \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}_v}})^D \xrightarrow{\sum \iota_v} \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}}})^D,$$

where $\iota_v : \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}_v}})^D \to \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}}})^D$ is the evident morphism.

Choosing any $h \in X$, the 1-cocycle

$$\Gamma_{\mathbf{R}} = \{1,c\} \to \mathbf{G}^{\mathrm{ad}}(\mathbf{C})$$

sending c to $\mathrm{ad}h(i)$ determines a class $[h] \in H^1(\mathbf{R},\mathbf{G}^{\mathrm{ad}})$ which is well-defined independently of h, and the associated inner form of $\mathbf{G}_{\mathbf{R}}$ is compact modulo center, cf. [XZ17, §2.1.3]. Moreover, the image of $[h]$ under $\alpha_{\mathbf{G}^{\mathrm{ad}},\infty}$ coincides with $\overline{\mu_h}_{\mathrm{ad}} \in \pi_0(Z(\widehat{\mathbf{G}^{\mathrm{ad}}})^{\Gamma_{\mathbf{R}}})^D$, where μ_h is the Hodge cocharacter associated with h, by [XZ17, §Lemma 2.1.4].

On the other hand, there is a natural commutative diagram of isomorphisms

$$\begin{tikzcd}
H^1(\mathbf{Q}_p,G^{\mathrm{ad}}) \arrow[r] \arrow[d, "\alpha_{G^{\mathrm{ad}},p}"] & B(G^{\mathrm{ad}})_{\mathrm{bas}} \arrow[d, "\kappa"] \\
\pi_0(Z(\widehat{G^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}_p}})^D \arrow[r] & X^*(Z(\widehat{G^{\mathrm{ad}}})^{\Gamma_{\mathbf{Q}_p}})
\end{tikzcd}$$

by [Kot85, Proposition 5.6 and Remark 5.7]. Let $b \in B(G,\mu_h^{-1})$ be the unique basic element, and b^{ad} its image in $B(G^{\mathrm{ad}})_{\mathrm{bas}}$. Then the inner form G_b defines a class $[G_b] \in H^1(\mathbf{Q}_p,G^{\mathrm{ad}})$ mapping to b^{ad} along the upper horizontal arrow. Moreover, since $b^{\mathrm{ad}} \in B(G^{\mathrm{ad}},\mu_{h,\mathrm{ad}}^{-1})$, the definition of the latter set implies that $\kappa(b^{\mathrm{ad}}) = \overline{\mu_h}_{\mathrm{ad}}^{-1}$. Going around the diagram, we conclude that $\alpha_{G^{\mathrm{ad}},p}([G_b]) = \overline{\mu_h}_{\mathrm{ad}}^{-1}$.

Now, let
$$\gamma = (\gamma_v) \in \oplus_{v \le \infty} H^1(\mathbf{Q}_v, \mathbf{G}^{\mathrm{ad}})$$
be the class defined by setting $\gamma_v = 0$ for all $v \notin \{p, \infty\}$, $\gamma_p = [G_b]$, and $\gamma_\infty = [h]$. The analysis in the previous two paragraphs shows that
$$\alpha(\gamma) = \overline{\mu_h}_{\mathrm{ad}} + \overline{\mu_h}_{\mathrm{ad}}^{-1} = 0,$$
so there is a unique class $[\mathbf{G}'] \in H^1(\mathbf{Q}, \mathbf{G}^{\mathrm{ad}})$ mapping to γ. It is clear by construction that the inner form $\mathbf{G}'$ has the desired properties. □

For any open compact subgroup $K \subset \mathbf{G}(\mathbf{A}^f)$, let $\mathcal{S}(\mathbf{G}, X)_K$ be the associated rigid analytic Shimura variety over $\mathbf{C}_p$ with level K. If $K^p \subset \mathbf{G}(\mathbf{A}^{f,p})$ is an open compact subgroup, set
$$\mathcal{S}(\mathbf{G}, X)_{K^p} = \varprojlim_{K_p \to \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}.$$
If $(\mathbf{G}, X)$ is of pre-abelian type, this is representable by a perfectoid space; in general it is just a diamond. By the results of [Han16], there is a canonical $\mathbf{G}(\mathbf{Q}_p)$-equivariant Hodge-Tate period map
$$\pi_{\mathrm{HT}} : \mathcal{S}(\mathbf{G}, X)_{K^p} \to \mathscr{F}\ell_{G,\mu}$$
of diamonds over $\operatorname{Spd} \mathbf{C}_p$, compatible with varying K^p, which recovers the construction of [CS17] for Hodge type Shimura varieties. By $\mathbf{G}(\mathbf{Q}_p)$-equivariance, π_{HT} descends to a map
$$\pi_{\mathrm{HT}, K_p} : \mathcal{S}(\mathbf{G}, X)_{K^p K_p} \to [\mathscr{F}\ell_{G,\mu} / \underline{K_p}]$$
for any open compact subgroup $K_p \subset \mathbf{G}(\mathbf{Q}_p)$.

Next, let $b \in B(G, \mu)$ be the unique basic element, and let $\mathscr{F}\ell^b_{G,\mu}$ be the basic Newton stratum in the flag variety as defined in §2.2. This is an open $G(\mathbf{Q}_p)$-stable subspace of $\mathscr{F}\ell_{G,\mu}$. By pullback along π_{HT}, this defines an open subspace $\mathcal{S}(\mathbf{G}, X)^b_{K^p} \subset \mathcal{S}(\mathbf{G}, X)_{K^p}$, which by $G(\mathbf{Q}_p)$-invariance descends to an open subspace $\mathcal{S}(\mathbf{G}, X)^b_K \subset \mathcal{S}(\mathbf{G}, X)_K$ for any $K \subset \mathbf{G}(\mathbf{A}^f)$.

Definition 3.2. Notation as above, we say a global Shimura datum $(\mathbf{G}, X)$ satisfies *basic uniformization at* p if there is a $\mathbf{G}(\mathbf{A}^f)$-equivariant isomorphism
$$\varprojlim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \overset{\psi}{\simeq} (\underline{\mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f)} \times_{\operatorname{Spd} \mathbf{C}_p} \mathrm{Sh}(G, \mu, b)_\infty) / \underline{G_b(\mathbf{Q}_p)} \qquad (\dagger)$$
of diamonds over $\operatorname{Spd} \mathbf{C}_p$ such that under the identification $\mathscr{F}\ell^b_{G,\mu} \cong \mathrm{Sh}(G, \mu, b)_\infty / \underline{G_b(\mathbf{Q}_p)}$, the Hodge-Tate period map
$$\pi_{\mathrm{HT}} : \varprojlim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \to \mathscr{F}\ell^b_{G,\mu}$$
identifies with the projection
$$(\underline{\mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f)} \times_{\operatorname{Spd} \mathbf{C}_p} \mathrm{Sh}(G, \mu, b)_\infty) / \underline{G_b(\mathbf{Q}_p)} \to \mathrm{Sh}(G, \mu, b)_\infty / \underline{G_b(\mathbf{Q}_p)}$$
onto the second factor. Moreover, we require that $\mathbf{G}(\mathbf{A}^f) \simeq \mathbf{G}'(\mathbf{A}^{f,p}) \times G(\mathbf{Q}_p)$ acts on the right-hand side of (†) via the natural actions $\mathbf{G}'(\mathbf{A}^{f,p}) \circlearrowright \mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f)$ and $G(\mathbf{Q}_p) \circlearrowright \mathrm{Sh}(G, \mu, b)_\infty$.

Let us make some remarks on this definition.

- This definition implicitly depends on the particular choices of isomorphisms $\mathbf{C} \simeq \overline{\mathbf{Q}_p}$, $\mathbf{G}'_{\mathbf{A}^{\{\infty p\}}} \simeq \mathbf{G}_{\mathbf{A}^{\{\infty p\}}}$, and $\mathbf{G}'_{\mathbf{Q}_p} \simeq G_b$, as well as on a choice of an actual element $b \in G(\breve{\mathbf{Q}}_p)$ representing the basic class in $B(G, \mu)$. In all cases where a Shimura datum is known to satisfy basic uniformization at p, the existence of an isomorphism ψ with the required properties is not sensitive to these choices (more precisely, the actual proofs that ψ exists are equivariant with respect to these choices).
- The quotient $\mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f)$ is naturally a profinite set, via the isomorphism
$$\mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f) \cong \varprojlim_{K \subset \mathbf{G}'(\mathbf{A}^f)} \mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f) / K$$
and the finiteness of the individual quotients $\mathbf{G}'(\mathbf{Q}) \backslash \mathbf{G}'(\mathbf{A}^f) / K$.

- The name "uniformization" is indeed appropriate: for any open compact subgroup $K^pK_p \subset \mathbf{G}(\mathbf{A}^f)$, passing to the quotients by K^pK_p on both sides of (†) induces an isomorphism
$$\mathcal{S}(\mathbf{G},X)^b_{K^pK_p} \simeq \coprod \mathrm{Sh}(G,\mu,b)_{K_p}/\Gamma_i$$
for some finite set of discrete subgroups $\Gamma_i \subset G_b(\mathbf{Q}_p)$.

Of course, we are not proposing this definition in a vacuum.

Theorem 3.3. *If $(\mathbf{G},X)$ is a Shimura datum of abelian type and $p>2$ is a prime such that $\mathbf{G}_{\mathbf{Q}_p}$ is unramified, then $(\mathbf{G},X)$ satisfies basic uniformization at p.*

Proof. In the Hodge type setting, Proposition 3.1 was proved by Xiao-Zhu [XZ17, Corollary 7.2.15], and the basic uniformization isomorphism then follows from work of Kim and Howard-Pappas [HP17, Kim18]. These results were then extended to the abelian type setting by Shen [She19]. □

One can also treat some cases of bad reduction; this already goes back to Rapoport-Zink's book.

Theorem 3.4. *Let $(\mathbf{G},X)$ be a PEL Shimura datum arising from a rational PEL datum $(F,B,*,V,\langle,\rangle,h)$ in Kottwitz's sense. Let p be a prime such that B contains a $\mathbf{Z}_{(p)}$-order $\mathcal{O}_B$ with $\mathcal{O}_B\otimes\mathbf{Z}_p$ maximal and $*$-stable in $B\otimes\mathbf{Q}_p$. Then $(\mathbf{G},X)$ satisfies basic uniformization at p.*

Proof. This follows from the analysis in [RZ96, Chapter 6]. Note that Rapoport-Zink in fact require the existence of a $\mathbf{Z}$-order in B with the specified properties, but this more restrictive condition is irrelevant and is actually not used anywhere in their arguments. □

Theorem 3.5. *Let (G,μ,b) be a basic local Shimura datum of the following type: assume that $G=\mathrm{Res}_{L/\mathbf{Q}_p}H$ where $H\simeq \mathrm{GL}_m(A)$ is an inner form of $\mathrm{GL}_{n/L}$ for some finite extension $L/\mathbf{Q}_p$, and that*
$$\mu:\mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}} = \prod_{\mathrm{Hom}(L,\overline{\mathbf{Q}_p})} \mathrm{GL}_{n,\overline{\mathbf{Q}_p}}$$
is as in the discussion preceding Theorem 1.5. Then we can find an isomorphism $\iota:\mathbf{C}\to\overline{\mathbf{Q}_p}$ and a PEL Shimura datum $(\mathbf{G},X)$ satisfying the conditions of Theorem 3.4 such that:

1. $\mathbf{G}_{\mathbf{Q}_p}\simeq G\times\mathbf{G}_m$*, and*
2. *$\iota\mu_h\times\mathrm{id}=\mu$ where μ_h is the (inverse) Hodge cocharacter associated with the given Shimura datum.*

In particular, the local Shimura varieties $\mathrm{Sh}(G,\mu,b)_K$ occur (up to a $\mathbf{G}_m$-factor) in the basic uniformization at p of the Shimura varieties associated with the datum $(\mathbf{G},X)$.

Proof. We loosely follow the analysis in [HT01, pp. 51-57]. Choose a totally real field $F^+/\mathbf{Q}$ such that $F^+\otimes\mathbf{Q}_p=L$ and an imaginary quadratic field E in which p is split. Let v be the unique place of F^+ above p. Set $F=F^+E$, and let w and w^c be the places of F above p, so $L\simeq F^+_v\simeq F_w$. Let B/F be a central division algebra of degree n^2 such that $B\otimes_{E,c}E\simeq B^{\mathrm{opp}}$, B_x splits for any place of F which is not split over F^+, and $B\otimes_F F_w\simeq M_m(A)$. Choose a positive involution $*$ of the second kind on B, and set $V=B$ regarded as a $B\otimes_F B^{\mathrm{op}}$-module in the natural way.

Any alternating $*$-Hermitian pairing $V\times V\to\mathbf{Q}$ is of the form $(x,y)_\beta=\mathrm{tr}_{B/\mathbf{Q}}(x\beta y^*)$ for some $\beta\in B^{*=-1}$. Any such pairing induces a perfect duality between $V\otimes_F F_w$ and $V\otimes_F F_{w^c}$. Fix a maximal order $\mathcal{O}\subset B_w$, and set
$$\Lambda_w=\mathcal{O}\subset B_w=V\otimes_F F_w=M_m(A)$$
Let $\Lambda_{w^c}\subset V\otimes_F F_{w^c}$ be the $\mathbf{Z}_p$-dual of Λ_w under the pairing $(,)_\beta:V\otimes\mathbf{Q}_p\times V\otimes\mathbf{Q}_p\to\mathbf{Q}_p$. Then $\Lambda=\Lambda_w\oplus\Lambda_{w^c}\subset V\otimes\mathbf{Q}_p$ is a $\mathbf{Z}_p$-lattice and the induced pairing $(,)_\beta:\Lambda\times\Lambda\to\mathbf{Z}_p$ is perfect. Now there is a unique maximal $*$-stable $\mathbf{Z}_{(p)}$-order $\mathcal{O}_B\subset B$ such that $\mathcal{O}_{B,w}=\mathcal{O}$, which by definition is the set of elements in B carrying Λ into itself under the natural B-action on $V\otimes\mathbf{Q}_p$.

For any β as in the discussion above, let $\mathbf{G}_\beta$ be the associated unitary similitude group over $\mathbf{Q}$ as defined in [HT01], and let $\mathbf{G}_{\beta,1}$ be the kernel of the similitude character. Note that the structure maps of these groups factor naturally over Spec F^+. Then $\mathbf{G}_{\beta,\mathbf{Q}_p} \simeq G \times \mathbf{G}_m$ for any such β, so it remains to check that for any tuple

$$\{(p_\tau, q_\tau) \in (\mathbf{Z}^2_{\geq 0})^{\mathrm{Hom}(F^+,\mathbf{R})}, p_\tau + q_\tau = n\},$$

the element β can be chosen such that $\mathbf{G}_{\beta,1} \times_{\mathrm{Spec}\, F^+,\tau} \mathrm{Spec}\,\mathbf{R} \simeq U(p_\tau, q_\tau)$ for all $\tau \in \mathrm{Hom}(F^+, \mathbf{R})$. Since we don't impose any conditions on $\mathbf{G}_{\beta,\mathbf{Q}_q}$ at any finite prime $q \neq p$, the existence of such a β follows from a much easier variant of the analysis in [HT01, pp. 52-55]. □

Remark 3.6. The $\mathbf{G}_m$ factor appearing in the previous theorem is indeed harmless. To explain this, let (G, μ, b) be any local Shimura datum. Consider a product local Shimura datum $(G \times T, \mu \times \mu_T, b \times b_T)$ where (T, μ_T, b_T) is any toral local Shimura datum. Let T_b be the inner form of T associated with b_T. Let ρ be any irreducible smooth $G_b(\mathbf{Q}_p)$-representation, and let χ be any smooth character of $T_b(\mathbf{Q}_p)$. Then a Künneth formula argument shows that

$$R\Gamma_c(G \times T, \mu \times \mu_T, b \times b_T)[\rho \boxtimes \chi] \cong R\Gamma_c(G, \mu, b)[\rho] \otimes R\Gamma_c(T, \mu_T, b_T)[\chi].$$

Moreover $R\Gamma_c(T, \mu_T, b_T)[\chi]$ is nonzero and concentrated in degree zero. Together with the compatibility of the construction $\rho \rightsquigarrow \varphi_\rho$ with products, this shows that Theorem 1.1 holds for the datum (G, μ, b) and all ρ satisfying condition 2. iff it holds for the datum $(G \times T, \mu \times \mu_T, b \times b_T)$ and all $\rho \boxtimes \chi$ satisfying condition 2. In particular, when proving Theorem 1.1, it is permissible to replace the datum (G, μ, b) by any product with a toral local Shimura datum.

It would be interesting to prove basic uniformization at p under the hypotheses of [KP18, Theorem 0.1]. This is probably within reach.

Corollary 3.7. *Let $(\mathbf{G}, X)$ be a Shimura datum satisfying basic uniformization at p as in Definition 3.2, and let $\mathbf{G}'$ be the associated inner form of $\mathbf{G}$. Fix some open compact subgroup $K^pK_p \subset \mathbf{G}(\mathbf{A}^f)$. Let $\mathcal{L}_{\xi,\mathcal{O}}$ be a K^p-stable $\mathcal{O}$-lattice in the E-linear realization of some irreducible algebraic representation of $\mathbf{G}$ of highest weight ξ, so (with the obvious abuse of notation) there is an associated ℓ-adic étale sheaf $\mathcal{L}_{\xi,\mathcal{O}}$ on $\mathcal{S}(\mathbf{G}, X)_{K^pK_p}$.*

Then there is a natural isomorphism

$$R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}|_{[\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]} \cong q^{b*}_{K_p}\mathcal{F}_\Pi$$

in $D_{\text{ét}}([\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}], \mathcal{O})$ compatibly with varying K_p. Here

$$q^b_{K_p} : [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}] \to B\underline{G_b(\mathbf{Q}_p)}$$

is the map defined in Proposition 2.16, and $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p, \mathcal{L}_{\xi,\mathcal{O}})^\wedge$ is the ℓ-adic completion of the space of algebraic automorphic forms defined in Proposition 2.9.

Note that since $\mathbf{G} \otimes_\mathbf{Q} \mathbf{Q}_\ell \simeq \mathbf{G}' \otimes_\mathbf{Q} \mathbf{Q}_\ell$, we can also naturally regard $\mathcal{L}_{\xi,\mathcal{O}}$ as a lattice in the E-linear realization of an algebraic representation of $\mathbf{G}'$, so the appeal to Proposition 2.9 makes sense.

Proof. Quotienting the basic uniformization isomorphism by K^pK_p, we get a Cartesian diagram

$$\begin{tikzcd}
\mathcal{S}(\mathbf{G}, X)^b_{K^pK_p} \arrow[r, "\pi^b_{\mathrm{HT},K_p}"] \arrow[d, "g"] & {[\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]} \arrow[d, "q^b_{K_p}"] \\
{[\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p}/\underline{G_b(\mathbf{Q}_p)}]} \arrow[r, "f"] & B\underline{G_b(\mathbf{Q}_p)}
\end{tikzcd}$$

where the horizontal maps are proper and the vertical maps are ℓ-cohomologically smooth. Moreover, the equivariance properties of the basic uniformization isomorphism directly imply that the restriction of $\mathcal{L}_{\xi,\mathcal{O}}$ to $\mathcal{S}(\mathbf{G}, X)^b_{K^pK_p}$ is the pullback along g of the correct local system (which we also denote by $\mathcal{L}_{\xi,\mathcal{O}}$) on the v-stack $[\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p}/\underline{G_b(\mathbf{Q}_p)}]$. The key point here is that $\mathcal{L}_{\xi,\mathcal{O}}$ in both instances is obtained by descending along the K^p-action on a suitable cover (where K^p acts on $\mathcal{L}_{\xi,E}$ via the evident map $K^p \to \mathbf{G}(\mathbf{Q}_\ell) \cong \mathbf{G}'(\mathbf{Q}_\ell)$), and the K^p-action on the right-hand side of the uniformization isomorphism (†) is concentrated entirely on the first factor.

With this in hand, we see that

$$R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}|_{[\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]} \cong R\pi^b_{\mathrm{HT},K_p*}g^*\mathcal{L}_{\xi,\mathcal{O}},$$

again with the obvious abuse of notation concerning the local systems. Moreover,

$$Rf_*\mathcal{L}_{\xi,\mathcal{O}} \in D_{\text{ét}}(B\underline{G_b(\mathbf{Q}_p)},\mathcal{O}) \cong \widehat{D}(G_b(\mathbf{Q}_p),\mathcal{O})$$

is admissible and concentrated in degree zero, and is given concretely as (the sheaf $\mathcal{F}_\Pi$ associated with) the claimed representation $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p,\mathcal{L}_{\xi,\mathcal{O}})^\wedge$. Since the vertical maps are smooth, the result now follows from the smooth base change isomorphism $q^{b*}_{K_p}Rf_* \cong R\pi^b_{\mathrm{HT},K_p*}g^*$. □

3.2. **Proof of Theorem 1.1.** In this section, we state and prove the key technical lemma, and conclude the proof of Theorem 1.1.

Lemma 3.8. *Let X be a quasicompact small v-stack with an ℓ-cohomologically smooth map $f : X \to \mathrm{Bun}_G$. Let $j : B\underline{G_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ be the inclusion of a basic stratum which meets the image of f, so we get a cartesian diagram*

$$\begin{tikzcd}
U \arrow[r, "\tilde{j}"] \arrow[d, "f^b"'] & X \arrow[d, "f"] \\
B\underline{G_b(\mathbf{Q}_p)} \arrow[r, "j"] & \mathrm{Bun}_G
\end{tikzcd}$$

where $\tilde{j}$ is an open immersion.

Let $\mathcal{F}$ be a complex in $D_{\text{ét}}(X,\mathcal{O})$. Assume that $\mathcal{F}|_U = f^{b*}\mathcal{G}_b$ for some complex $\mathcal{G}_b \in D_{\text{ét}}(B\underline{G_b(\mathbf{Q}_p)},\mathcal{O})$. Let $\rho^\circ \in \mathrm{Rep}_{\mathcal{O}}(G_b(\mathbf{Q}_p))_{\mathrm{adm}}$ be an inert representation of $G_b(\mathbf{Q}_p)$, and let $\mathcal{F}_{\widehat{\rho^\circ}} \in D_{\text{ét}}(B\underline{G_b(\mathbf{Q}_p)},\mathcal{O})$ be the associated ℓ-adic sheaf.

If $\mathcal{F}_{\widehat{\rho^\circ}}$ occurs as an isogeny direct summand of $\mathcal{G}_b$, then $f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \cong \tilde{j}_!f^{b*}\mathcal{F}_{\widehat{\rho^\circ}}$ occurs as an isogeny direct summand of $\mathcal{F}$ in $D_{\text{ét}}(X,\mathcal{O})$.

The essential strength of this lemma is that we don't need to know anything about the complementary restriction $\mathcal{F}|_{X\smallsetminus U}$.

Proof. Let $i : \mathrm{Bun}_G \smallsetminus B\underline{G_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ be the inclusion of the closed complement, and let $\tilde{i} : Z \to X$ be the pullback of i.

By assumption, we can pick an isogeny inclusion $\mathcal{F}_{\widehat{\rho^\circ}} \to \mathcal{G}_b$, which pulls back to an isogeny inclusion $f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \to f^*j_!\mathcal{G}_b = \tilde{j}_!\tilde{j}^*\mathcal{F}$. These can be embedded in a larger diagram

$$\begin{tikzcd}
f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \arrow[d] \arrow[dr] & & & \\
\tilde{j}_!\tilde{j}^*\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[dr] & \tilde{i}_*\tilde{i}^*\mathcal{F} \arrow[r] & {} \\
K \arrow[d] & & L \arrow[dr] & \\
{} & & & {}
\end{tikzcd}$$

where by definition the small triangle commutes and each of the three sets of collinear arrows forms a distinguished triangle. By Proposition 2.2.3, it suffices to show that the map $h : L \to f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]$ is killed by power of ℓ.

By the octahedral axiom, L sits in a distinguished triangle $K \to L \to \tilde{i}_*\tilde{i}^*\mathcal{F} \to$. Applying $\mathrm{Hom}(-, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1])$ to this triangle gives an exact sequence

$$\mathrm{Hom}(\tilde{i}_*\tilde{i}^*\mathcal{F}, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]) \xrightarrow{\alpha} \mathrm{Hom}(L, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]) \xrightarrow{\beta} \mathrm{Hom}(K, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]).$$

The image of h under the map β is the connecting map $K \to f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]$ associated with the vertical triangle, which is killed by a power of ℓ since the map $f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \to \tilde{j}_!\tilde{j}^*\mathcal{F}$ is an isogeny inclusion (using Proposition 2.2.3 again). Therefore $\ell^n h \in \mathrm{im}\alpha$ for some n, so it suffices to show that $\mathrm{Hom}(\tilde{i}_*\tilde{i}^*\mathcal{F}, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1])$ is killed by a power of ℓ.

In fact, we claim that $R\mathrm{Hom}(\tilde{i}_*\tilde{i}^*\mathcal{F}, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}})$ is killed by a power of ℓ. For this, note that $f^*(-) = \mathcal{L} \otimes Rf^!(-)$ for some shifted rank one local system $\mathcal{L}$, since f is ℓ-cohomologically smooth. Then

$$\begin{aligned} R\mathrm{Hom}(\tilde{i}_*\tilde{i}^*\mathcal{F}, f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}) &\cong R\mathrm{Hom}(\tilde{i}_*\tilde{i}^*\mathcal{F}, \mathcal{L} \otimes Rf^!j_!\mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(Rf_!(\tilde{i}_*\tilde{i}^*\mathcal{F} \otimes \mathcal{L}^{-1}), j_!\mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(i_*i^*Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), j_!\mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(i^*Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}}) \end{aligned}$$

where the second and fourth lines follow from adjointness and the third line follows from two applications of proper base change. Quite generally we have $Ri^!j_!(-) = i^*Rj_*(-)[-1]$, so $Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}} \cong i^*Rj_*\mathcal{F}_{\widehat{\rho^\circ}}[-1]$. Since ρ° is inert by assumption, the cone C of the natural map $j_!\mathcal{F}_{\widehat{\rho^\circ}} \to Rj_*\mathcal{F}_{\widehat{\rho^\circ}}$ is killed by a power of ℓ after restriction to any quasicompact open substack $W \subset \mathrm{Bun}_G$. Applying i^* to this map and noting that $i^*j_! = 0$, we deduce that $Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}} \cong i^*Rj_*\mathcal{F}_{\widehat{\rho^\circ}}[-1] \cong i^*C[-1]$ is killed by a power of ℓ after restriction to any such W. Choosing W large enough to contain the image of f, the natural map

$$R\mathrm{Hom}(i^*Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}}) \to R\mathrm{Hom}(i^*Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1})|_W, Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}}|_W)$$

is an isomorphism. Since $Ri^!j_!\mathcal{F}_{\widehat{\rho^\circ}}|_W$ is killed by a power of ℓ, this concludes the proof. □

We can now prove our main result.

Proof of Theorem 1.1. Fix (G, μ, b) as in the statement of the theorem, and let ρ be an inert supercuspidal representation ρ of $G_b(\mathbf{Q}_p)$. Set $d = \dim \mathrm{Sh}(G, \mu, b)_K$. Fix a global Shimura datum $(\mathbf{G}, X)$ such that $\mathrm{Sh}(G, \mu, b)_K$ occurs in the associated basic uniformization at p. Note that the Shimura varieties associated with this Shimura datum also have dimension d. Let $\mathbf{G}'$ be the inner form of $\mathbf{G}$ occurring in the basic uniformization. After replacing ρ by an unramified twist if necessary, Proposition 2.9 and Theorem 2.22 imply that we may choose an inert $\mathcal{O}$-lattice $\rho^\circ \subset \rho$, together with some K^p and $\mathcal{L}_{\xi,\mathcal{O}}$ as in the statement of Proposition 2.9 such that ρ° occurs as an isogeny direct summand of $\mathcal{A}_{\mathbf{G}'}(K^p, \mathcal{L}_{\xi,\mathcal{O}})$. Let $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p, \mathcal{L}_{\xi,\mathcal{O}})^\wedge$ be the ℓ-adic completion, so $\widehat{\rho^\circ}$ is an isogeny direct summand of Π.

We are going to apply Lemma 3.8 to $X = [\mathscr{F}\ell_{G,\mu}/\underline{K_p}]$ and the map $f = q_{K_p} : [\mathscr{F}\ell_{G,\mu}/\underline{K_p}] \to \mathrm{Bun}_G$, with $\mathcal{F} = R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}$ and with $b \in B(G, \mu)$ the unique basic element. Then $U = [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]$, and Corollary 3.7 implies that $\mathcal{F}|_U \cong q^{b*}_{K_p}\mathcal{G}_b$ upon setting $\mathcal{G}_b = \mathcal{F}_\Pi$. Moreover, $\widehat{\rho^\circ}$ is an isogeny direct summand of Π by design, so $\mathcal{F}_{\widehat{\rho^\circ}}$ is an isogeny direct summand of $\mathcal{F}_\Pi = \mathcal{G}_b$.

This verifies the hypotheses of Lemma 3.8. Applying that lemma, we deduce that there is an isogeny inclusion

$$\tilde{j}_!q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}} \to R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}$$

functorially in K_p, where

$$\tilde{j} : [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}] \to [\mathscr{F}\ell_{G,\mu}/\underline{K_p}]$$

is the natural open immersion. Applying $R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}], -)$, we deduce that there is an isogeny inclusion

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}], \tilde{j}_!q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}}) \to R\Gamma(\mathcal{S}(\mathbf{G}, X)_{K^pK_p}, \mathcal{L}_{\xi,\mathcal{O}})$$

functorially in K_p. In particular, inverting ℓ and applying Proposition 2.17, we conclude that

$$R\Gamma_c(\mathrm{Sh}(G,\mu,b)_{K_p},E)\otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))}\rho\cong R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}],\tilde{j}_! q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}})[1/\ell]$$

is a direct summand of $R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^pK_p},\mathcal{L}_{\xi,E})$. On the other hand, ξ is regular, so $H^i(\mathcal{S}(\mathbf{G},X)_{K^pK_p},\mathcal{L}_{\xi,E})=0$ for all $i<\dim\mathcal{S}(\mathbf{G},X)_K=d$ by Proposition 3.9 below. Since

$$H^i_c(G,\mu,b)[\rho]=\mathrm{colim}_{K_p}H^i(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_{K_p},E)\otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))}\rho),$$

we conclude that $H^i_c(G,\mu,b)[\rho]=0$ for all $i<d$. Rerunning the entire argument with ρ replaced by ρ^* and applying Theorem 2.23, we deduce also that

$$H^i_c(G,\mu,b)[\rho]^*\cong H^{2d-i}_c(G,\mu,b)[\rho^*](d)=0$$

for all $i>d$. This completes the proof. □

The key source of vanishing in this argument is the following result, which is essentially due to Li-Schwermer.

Proposition 3.9. *Let $(\mathbf{G},X)$ be any Shimura datum, and let $\mathcal{L}_{\xi,E}$ be the E-linear realization of an irreducible algebraic representation of $\mathbf{G}$ of regular highest weight. Then $H^i(\mathcal{S}(\mathbf{G},X)_{K^pK_p},\mathcal{L}_{\xi,E})=0$ for all $i<\dim\mathcal{S}(\mathbf{G},X)_K$.*

Proof. Writing $\mathrm{Sh}(\mathbf{G},X)_{K^pK_p}$ for the usual complex Shimura variety, the analogous vanishing $H^i(\mathrm{Sh}(\mathbf{G},X)_{K^pK_p},\mathcal{L}_{\xi,\mathbf{C}})=0$ for all $i<\dim\mathrm{Sh}(\mathbf{G},X)_K$ follows from the main theorems of [LS04]. Combining this with the Artin comparison theorem, the invariance of étale cohomology under chage of algebraically closed base field, and the comparison theorems in [Hub96, Chapter 3], we get the desired result. □

3.3. **Remaining proofs.** In this section we prove the remaining results stated in the introduction.

Proof of Theorems 1.4 and 1.6. We first prove Theorem 1.4. Combining Theorem 1.1 and [HKW22, Theorem 1.0.2], we deduce that $H^i_c(G,\mu,b)[\rho]=0$ for all $i\neq d$, and hence the equality

$$\left[H^d_c(G,\mu,b)[\rho]\right]=\sum_{\pi\in\Pi_\phi(G)}\dim\mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho},r_\mu)[\pi]$$

in $\mathrm{Groth}(G(\mathbf{Q}_p))$, which we can reinterpret as an isomorphism

$$H^d_c(G,\mu,b)[\rho]\simeq^{\mathrm{ss}}\oplus_{\pi\in\Pi_\phi(G)}\dim\mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho},r_\mu)\cdot\pi$$

where $\simeq^{\mathrm{ss}}$ denotes an isomorphism of semisimplified $G(\mathbf{Q}_p)$-representations. Let ω be the central character of ρ. A simple computation shows that $H^d_c(G,\mu,b)[\rho]$, and hence any subquotient, also has central character ω under the canonical identification $Z=Z_G(\mathbf{Q}_p)\cong Z_{J_b}(\mathbf{Q}_p)$; the key point here is that under this identification of centers, the diagonal action of $Z\subset Z\times Z\subset G(\mathbf{Q}_p)\times G_b(\mathbf{Q}_p)$ on $\mathrm{Sh}(G,\mu,b)_\infty$ is trivial. This shows that $H^d_c(G,\mu,b)[\rho]$ is an iterated extension of the elements $\pi\in\Pi_\phi(G)$ (with multiplicities as given above) in the category $\mathrm{Rep}_E(G(\mathbf{Q}_p))_\omega$. Since all elements $\pi\in\Pi_\phi(G)$ are supercuspidal and any supercuspidal representation with central character ω is both projective and injective as an object of the category $\mathrm{Rep}_E(G(\mathbf{Q}_p))_\omega$ [AR04], all the relevant extension classes are zero. Therefore $H^d_c(G,\mu,b)[\rho]$ is semisimple, so

$$H^d_c(G,\mu,b)[\rho]=\oplus_{\pi\in\Pi_\phi(G)}\dim\mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho},r_\mu)\cdot\pi$$

as desired.

Theorem 1.6 now follows, since the Fargues-Scholze construction agrees with the known local Langlands correspondence for inner forms of restrictions of scalars of GL_n [HKW22, Theorem 1.0.3]. This translates condition 2. of Theorem 1.3 into the stated condition on ρ. Condition 1. is a consequence of Theorem 3.5 and Remark 3.6. □

Proof of Theorem 1.5. Combine Theorem 1.1 with [Far04, Théorème 8.1.4] or [Shi12b, Corollary 1.3]. Again, Condition 1. follows from Theorem 3.5 and Remark 3.6. □

Before proving Theorem 1.8, let us define $\mathscr{C}_\varphi$ in general. For this, fix a reductive group $G/\mathbf{Q}_p$ and a semisimple L-parameter $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$. Recall from [FS21, Theorem IX.4.2] that there is a natural map $\psi : \mathcal{Z}^{\mathrm{spec}}(G)_{\overline{\mathbf{Q}_\ell}} \to \mathcal{Z}(D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell}))$. On the other hand, φ defines a closed point in the coarse moduli space of L-parameters, and hence a maximal ideal $\mathfrak{m}_\varphi \subset \mathcal{Z}^{\mathrm{spec}}(G)_{\overline{\mathbf{Q}_\ell}}$.

Definition 3.10. Notation as above, we write $\mathscr{C}_\varphi \subset D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ for the full subcategory spanned by objects A such that the induced map $\psi_A : \mathcal{Z}^{\mathrm{spec}}(G)_{\overline{\mathbf{Q}_\ell}} \to \mathrm{End}(A)$ factors over the $\mathfrak{m}_\varphi$-adic completion of $\mathcal{Z}^{\mathrm{spec}}(G)_{\overline{\mathbf{Q}_\ell}}$.

One can check that $\mathscr{C}_\varphi \subset D_{\mathrm{lis}}$ is a triangulated category stable under direct sums and Hecke operators, and that $\mathrm{Hom}(\mathscr{C}_\varphi, \mathscr{C}_{\varphi'}) = 0$ for any $\varphi' \neq \varphi$.

Proof of Theorem 1.8. Fix a finite extension $L/\mathbf{Q}_p$ and an embedding $L \to \overline{\mathbf{Q}_p}$. Set $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(L, \overline{\mathbf{Q}_p})$. Let $G = \mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ be as in the theorem, and fix a supercuspidal L-parameter $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$. Note that by Proposition 8.4 in [Bor79], isomorphism classes of such φ's are in bijection (in an obvious abuse of notation) with isomorphism classes of continuous irreducible $\varphi : W_L \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$. Note also that $\widehat{G} \cong \mathrm{GL}_{n,\overline{\mathbf{Q}_\ell}}^\Sigma$.

First, we note that $\mathscr{C}_\varphi$ can be described extremely explicitly in this situation. To give this description, recall that under the canonical identification $B(\mathrm{GL}_n, L) = B(G)_{\mathrm{bas}} \cong \mathbf{Z}$ given by sending an isocrystal to its degree, the composite map

$$\mathbf{Z} \cong B(G)_{\mathrm{bas}} \to H^1(\mathbf{Q}_p, G^{\mathrm{ad}})$$

sends d to the inner form $G_d = \mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_m(D)$, where $m = \gcd(d,n)$ and D is the central division algebra over L with Hasse invariant $= \frac{d}{n} \bmod 1$. In this notation, we have a canonical identification

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathrm{ss}}, \overline{\mathbf{Q}_\ell}) \cong \prod_{d \in \mathbf{Z}} D_{\mathrm{lis}}(B\underline{G_d(\mathbf{Q}_p)}, \overline{\mathbf{Q}_\ell}) = \prod_{d \in \mathbf{Z}} D(G_d(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell}).$$

For each d, let $\pi_{\varphi,d}$ be the unique irreducible supercuspidal representation of $G_d(\mathbf{Q}_p)$ with L-parameter φ, and let $\mathcal{F}_{\pi_{\varphi,d}}$ be the associated sheaf on $B\underline{G_d(\mathbf{Q}_p)}$ as before. Then an arbitrary object

$$\mathcal{F} = \prod \mathcal{F}_d \in D_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathrm{ss}}, \overline{\mathbf{Q}_\ell})$$

lies in $\mathscr{C}_\varphi$ if and only if $H^i(\mathcal{F}_d)$ is an iterated self-extension of copies of $\mathcal{F}_{\pi_{\varphi,d}}$ for all d and all i.

Next we check that T_W is t-exact on $\mathscr{C}_\varphi$ for any irreducible minuscule $W \in \mathrm{Rep}(\widehat{G})$. For any irreducible representation, $(\overline{\mathbf{Q}_\ell}^\times)^\Sigma \cong Z_{\widehat{G}}(\overline{\mathbf{Q}_\ell})$ acts on W via scaling by a character $(x_\sigma)_{\sigma \in \Sigma} \mapsto \prod x_\sigma^{d_\sigma}$. Set $d(W) = \sum d_\sigma$. Assuming W is minuscule and replacing W by a central twist, we can assume that the cocharacter μ_W corresponding to the highest weight character is as in the discussion before Theorem 1.5. Then unwinding all definitions and appealing to Theorem 1.6, we deduce that

$$T_W \mathcal{F}_{\pi_{\varphi,d}} = \mathcal{F}_{\pi_{\varphi,d+d(W)}}^{\oplus \dim W}$$

as elements of $\mathscr{C}_\varphi$. By the discussion in the previous paragraph, this proves the t-exactness of any such T_W. Since any representation is a summand of a tensor product of minuscule representations, we now conclude t-exactness of general Hecke operators by geometric Satake. Now, set $\mathscr{F}_\varphi = \prod_{d \in \mathbf{Z}} \mathcal{F}_{\pi_{\varphi,d}}$. Then $T_W \mathscr{F}_\varphi \simeq \mathscr{F}_\varphi^{\oplus \dim W}$ for any minuscule W.

We give the remainder of the proof in the case where $L = \mathbf{Q}_p$; the general case is entirely analogous. Let $\mathscr{G}_\varphi$ be the Hecke eigensheaf constructed by Anschütz-Le Bras [ALB21]. Writing $\mathscr{G}_{\varphi,d} = \mathscr{G}_\varphi | B\underline{G_d(\mathbf{Q}_p)}$, we need to prove that $\mathscr{G}_{\varphi,d} \simeq \mathcal{F}_{\pi_{\varphi,d}}$ for all d. By the analysis in [ALB21], we know that $\mathscr{G}_{\varphi,1} \simeq \mathcal{F}_{\pi_{\varphi,1}}$. Applying T_{Std} to this isomorphism and forgetting the Weil group action, the arguments in the previous paragraph show that

$$T_{\mathrm{Std}} \mathscr{G}_{\varphi,1} \simeq T_{\mathrm{Std}} \mathcal{F}_{\pi_{\varphi,1}} \simeq \mathcal{F}_{\pi_{\varphi,2}}^{\oplus n}.$$

On the other hand, $T_{\mathrm{Std}} \mathscr{G}_{\varphi,1} \simeq \mathscr{G}_{\varphi,2} \boxtimes \varphi$ by the Hecke eigensheaf property. Taken together, these observations show that $\mathscr{G}_{\varphi,2} \boxtimes \varphi \simeq \mathcal{F}_{\pi_{\varphi,2}} \boxtimes \varphi'$ for some n-dimensional $W_{\mathbf{Q}_p}$-representation φ'. Using the irreducibility of $\pi_{\varphi,2}$ and φ, this forces isomorphisms $\mathscr{G}_{\varphi,2} \simeq \mathcal{F}_{\pi_{\varphi,2}}$ and $\varphi \simeq \varphi'$. Applying T_{Std} to the isomorphism $\mathscr{G}_{\varphi,2} \simeq \mathcal{F}_{\pi_{\varphi,2}}$

and repeating this argument, we conclude by induction that $\mathscr{G}_{\varphi,j} \simeq \mathcal{F}_{\pi_{\varphi,j}}$ for all $j \geq 1$. For $j < 1$, we instead apply $T_{\mathrm{Std}^\vee}$ to the isomorphism $\mathscr{G}_{\varphi,1} \simeq \mathcal{F}_{\pi_{\varphi,1}}$ and argue by downwards induction. □

Remark 3.11. By combining the arguments in [ALB21] with [HKW22, Theorem 1.0.2], it is not difficult to see that $\mathscr{G}_{\varphi,j} \simeq \mathcal{F}_{\pi_{\varphi,j}}[n_j]$ for all $j \in \mathbf{Z}$, but with some unspecified shifts n_j. When φ is self-dual, a simple duality argument shows that $n_j = 0$. However, for a general φ, it doesn't seem possible to show that $n_j = 0$ for all $j \in \mathbf{Z}$ by some purely formal argument. One really needs something like Theorem 1.1.

Proof of Theorem 1.9. The only thing which requires explanation here is the definition of the "intersection cohomology" $IH_c^i(G, \mu, b)[\rho]$. For this, one simply repeats the definition of $H_c^i(G, \mu, b)[\rho]$ from the minuscule case, but with the constant sheaf $\mathcal{O}/\varpi^n$ on a qc open subspace $U \subset \mathrm{Sh}(G, \mu, b)_K$ replaced by the pullback along the Grothendieck-Messing period map of the sheaf $\mathcal{S}_{V_\mu, \mathcal{O}/\varpi^n} \in D(\mathrm{Gr}_{G,\leq\mu}, \mathcal{O}/\varpi^n)$ associated with the Weyl module V_μ by geometric Satake (cf. [HKW22, §2.4]). Note that in the minuscule case, $IH_c^i(G, \mu, b)[\rho] = H_c^{i+d}(G, \mu, b)[\rho](\frac{d}{2})$.

The result now follows from unwinding all definitions. In particular, one checks that

$$R\Gamma_c(G, \mu, b, \mathcal{IC}_{V_\mu}) \otimes^{\mathbf{L}}_{\mathcal{H}(G_b(\mathbf{Q}_p))} \rho \cong (T_{V_\mu^\vee} j_! \mathcal{F}_\rho)|B\underline{G(\mathbf{Q}_p)}.$$

□

References

[ALB21] Johannes Anschütz and Arthur-César Le Bras, *Averaging functors in Fargues' program for* GL_n, preprint.

[AR04] Jeffrey D. Adler and Alan Roche, *Injectivity, projectivity and supercuspidal representations*, J. London Math. Soc. (2) **70** (2004), no. 2, 356–368. MR 2078898

[Art89] James Arthur, *Unipotent automorphic representations: conjectures*, no. 171-172, 1989, Orbites unipotentes et représentations, II, pp. 13–71. MR 1021499

[BMHN22] Alexander Bertoloni-Meli, Linus Hamann, and Kieu Hieu Nguyen, *Compatibility of the Fargues-Scholze correspondence for unitary groups*, preprint.

[BMY20] Alexander Bertoloni Meli and Alex Youcis, *An approach to the characterization of the local Langlands correspondence*, preprint.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61. MR 546608

[Boy09] Pascal Boyer, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. **177** (2009), no. 2, 239–280. MR 2511742

[Car90] H. Carayol, *Nonabelian Lubin-Tate theory*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 15–39. MR 1044827

[CS17] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. MR 3702677

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117. MR 771672

[Fal02] Gerd Faltings, *A relation between two moduli spaces studied by V. G. Drinfeld*, Algebraic number theory and algebraic geometry, Contemp. Math., vol. 300, Amer. Math. Soc., Providence, RI, 2002, pp. 115–129. MR 1936369

[Far04] Laurent Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, no. 291, 2004, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 1–199. MR 2074714

[Far15] ———, *G-torseurs en theorie de Hodge p-adique*, preprint.

[Far16a] ———, *Dualité de Poincaré et involution de Zelevinsky dans la cohomologie étale équivariante des espaces analytiques rigides*, preprint.

[Far16b] ———, *Geometrization of the local Langlands correspondence: an overview*, preprint.

[FS21] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, preprint.

[GT11] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. MR 2800725

[Ham21] Linus Hamann, *Compatibility of the Gan-Takeda and Fargues-Scholze local Langlands*, preprint.

[Han16] David Hansen, *Period morphisms and variations of p-adic Hodge structure*, preprint.

[Han17] ———, *Degenerating vector bundles in p-adic Hodge theory*, preprint.

[Han20] ———, *Vanishing and comparison theorems in rigid analytic geometry*, Compos. Math. **156** (2020), no. 2, 299–324. MR 4045974

[Har97] Michael Harris, *Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol's program*, Invent. Math. **129** (1997), no. 1, 75–119. MR 1464867

[HK19] Paul Hamacher and Wansu Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), no. 3-4, 973–1044. MR 4023369

[HKW22] David Hansen, Tasho Kaletha, and Jared Weinstein, *On the Kottwitz conjecture for local shtuka spaces*, Forum Math. Pi **10** (2022), Paper No. e13, 79. MR 4430954

[HP17] Benjamin Howard and Georgios Pappas, *Rapoport-Zink spaces for spinor groups*, Compos. Math. **153** (2017), no. 5, 1050–1118. MR 3705249

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[Hub96] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 1734903

[Ito13] Tetsushi Ito, *Supercuspidal representations in the cohomology of the Rapoport-Zink space for the unitary group in three variables*, RIMS Kokyuroku No. 1871 pp. 105-116.

[Kal16] Tasho Kaletha, *The local Langlands conjectures for non-quasi-split groups*, Families of automorphic forms and the trace formula, Simons Symp., Springer, [Cham], 2016, pp. 217–257. MR 3675168

[Kim18] Wansu Kim, *Rapoport-Zink uniformization of Hodge-type Shimura varieties*, Forum Math. Sigma **6** (2018), e16, 36. MR 3849611

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. MR 809866

[Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR 858284

[Kot90] ———, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR 1044820

[KP18] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218. MR 3905466

[LS04] Jian-Shu Li and Joachim Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169. MR 2060025

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur16] ———, *Higher algebra*, preprint.

[Man04] Elena Mantovan, *On certain unitary group Shimura varieties*, no. 291, 2004, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 201–331. MR 2074715

[Mie10] Yoichi Mieda, *Non-cuspidality outside the middle degree of ℓ-adic cohomology of the Lubin-Tate tower*, Adv. Math. **225** (2010), no. 4, 2287–2297. MR 2680204

[Mie21] ———, *Local Saito-Kurokawa A-packets and ℓ-adic cohomology of Rapoport-Zink tower for* $\mathrm{GSp}(4)$*: announcement*, preprint.

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263

[Rog83] Jonathan D. Rogawski, *Representations of* $\mathrm{GL}(n)$ *and division algebras over a p-adic field*, Duke Math. J. **50** (1983), no. 1, 161–196. MR 700135

[RV14] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326. MR 3271247

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439

[Sch17] Peter Scholze, *Étale cohomology of diamonds*, preprint.

[She19] Xu Shen, *On some generalized Rapoport-Zink spaces*, preprint.

[Shi12a] Sug Woo Shin, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), no. 1, 83–120. MR 3004076

[Shi12b] ———, *On the cohomology of Rapoport-Zink spaces of EL-type*, Amer. J. Math. **134** (2012), no. 2, 407–452. MR 2905002

[Sta20] The Stacks Project Authors, *stacks project*, `http://stacks.math.columbia.edu`, 2020.

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-Adic geometry*, Annals of Math. Studies.

[Vig04] Marie-France Vignéras, *On highest Whittaker models and integral structures*, Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 773–801. MR 2058628

[XZ17] Liang Xiao and Xinwen Zhu, *Cycles on Shimura varieties via geometric Satake*, preprint.

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