The Petersson norm of the Jacobi theta function

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Let $\theta(z) = \sum_{n\in\mathbf{Z}} e^{2\pi i n^2 z}$ be the Jacobi theta function. This is a modular form of weight 1/2 for the group $\Gamma_0(4)$, and is well-known to be square-integrable; in fact, it's the first interesting non-cuspidal but square-integrable automorphic form. In this note we compute the norm

$$\|\theta\|^2 := \int_{\Gamma_0(4)\backslash\mathfrak{H}} y^{\frac{1}{2}} |\theta(z)|^2 \frac{dxdy}{y^2}.$$

Theorem. *The Petersson norm of* θ *is* $\|\theta\|^2 = 4\pi$.

Rather surprisingly, I have never seen this number calculated anywhere, and I have seen at least one prominent researcher introduce it as a kind of "fundamental constant" in a paper. The problem is that the constant term of θ prevents one from immediately realizing $\|\theta\|^2$ as the residue of a Rankin-Selberg style integral. We get around this by a little trick.

Fix an arbitrary odd prime p, and consider the integral

$$I_p(s) = \int_{[0,1]\times\mathbf{R}_{>0}} y^{s+\frac{1}{2}} \left(|\theta(z)|^2 - |\theta(p^2 z)|^2\right) \frac{dxdy}{y^2}.$$

This converges absolutely for $\mathrm{Re} s > 1$ and is easily calculated as

$$\begin{aligned} I_p(s) &= 2\int_{\mathbf{R}_{>0}} y^{s-1/2} \sum_{n\geq 1,\, p\nmid n} e^{-4\pi n^2 y} \frac{dy}{y} \\ &= 2\cdot(4\pi)^{1/2-s} \sum_{n\geq 1,\, p\nmid n} n^{1-2s} \int_{\mathbf{R}_{>0}} y^{s-1/2} \frac{dy}{y} \\ &= 2\cdot(4\pi)^{1/2-s}\Gamma(s-\tfrac{1}{2})(1-p^{1-2s})\zeta(2s-1). \end{aligned}$$

On the other hand, the function $y^{\frac{1}{2}}\left(|\theta(z)|^2 - |\theta(p^2 z)|^2\right)$ is invariant under the group $\Gamma_0(4p^2)$, so folding up gives

$$I_p(s) = \int_{\Gamma_0(4p^2)\backslash\mathfrak{H}} E_{4p^2}(z,s) y^{\frac{1}{2}} \left(|\theta(z)|^2 - |\theta(p^2 z)|^2\right) d\mu(z),$$

where $E_{4p^2}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4p^2)} \mathrm{Im}(\gamma z)^s$ is the usual nonholomorphic Eisenstein series and $d\mu(z) = \frac{dxdy}{y^2}$. This series has a simple pole at $s=1$ with residue $\frac{3}{\pi} \cdot [\Gamma_0(1) : \Gamma_0(4p^2)]^{-1} = \frac{1}{2p(p+1)\pi}$. Hence taking residues gives

$$\begin{aligned}
\mathrm{res}_{s=1} I_p(s) &= \frac{1}{2p(p+1)\pi} \int_{\Gamma_0(4p^2)\backslash \mathfrak{H}} y^{\frac{1}{2}} \left(|\theta(z)|^2 - |\theta(p^2 z)|^2 \right) d\mu(z) \\
&= \frac{1}{2\pi} \int_{\Gamma_0(4)\backslash \mathfrak{H}} y^{\frac{1}{2}} |\theta(z)|^2 d\mu(z) - \frac{1}{2p(p+1)\pi} \int_{\Gamma_0(4p^2)\backslash \mathfrak{H}} y^{\frac{1}{2}} |\theta(p^2 z)|^2 d\mu(z) \\
&= \frac{1}{2\pi} \|\theta\|^2 - \frac{1}{2p^2(p+1)\pi} \int_{\Gamma_0(4p^2)\backslash \mathfrak{H}} y^{\frac{1}{2}} |\theta(z)|^2 d\mu(z) \\
&= \frac{1}{2\pi} (1 - p^{-1}) \|\theta\|^2 ,
\end{aligned}$$

where the third line follows from changing variables in the second integral via the involution $z \to \frac{-1}{4p^2 z}$ and the transformation law $\theta(\frac{-1}{4z}) = \sqrt{\frac{2z}{i}}\theta(z)$. But our first computation gives

$$\mathrm{res}_{s=1} I_p(s) = 2(1 - p^{-1}),$$

and p was arbitrary, so $\|\theta\|^2 = 4\pi$. $\square$