Notes on ULA complexes

David Hansen

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In §5.1 of Braverman and Gaitsgory's paper "Geometric Eisenstein series" [BG], the authors define what it means for an ℓ-adic complex on a variety X to be *f-(universally) locally acyclic* with respect to a morphism $f : X \to Y$. Although the definition appears somewhat technical at first, complexes satisfying this condition enjoy many remarkable properties, recorded in §5.1.2 of [BG]. Since then, f-LA and f-ULA complexes have grown to play a modest but respectable role in the geometric Langlands program (cf. [R], [Ri], etc.) Unfortunately, Braverman and Gaitsgory omit the proofs of their basic properties, some of which are not at all obvious, and it took me some time to figure out why these claimed properties are actually true. Here I record what I worked out, in the hope it may help someone else trying to understand this material.

Fix a base field k, which for simplicity I'll assume is algebraically closed; in particular, I will ignore all Tate twists in what follows. By "variety" I mean a separated k-scheme of finite type. Fix a coefficient ring $\Lambda = \overline{\mathbf{Q}_\ell}$ for some ℓ invertible in k. For any variety X, I abbreviate $D(X) = D(X_{\text{ét}}, \Lambda)$. For a variety X with structure map $\pi : X \to \operatorname{Spec} k$ set $\omega_X = R\pi^!\Lambda$, and write $\mathbf{D}\mathcal{F}$ for the Verdier dual $R\mathscr{H}\mathrm{om}(\mathcal{F}, \omega_X)$. We frequently use the fact that Verdier duality exchanges $Rf_!$ with Rf_*, and f^* with $Rf^!$, as well as the biduality isomorphism $\mathcal{F} \overset{\sim}{\to} \mathbf{DD}\mathcal{F}$ for any $\mathcal{F} \in D_c(X)$.

We begin by recalling the definition of f-(U)LA complexes (which is presented in [BG] as a terse series of non-sequiturs). Recall that for any separated finite type map $g : T \to S$ and any $\mathcal{F}, \mathcal{G} \in D(S)$, there is a natural map $\gamma : g^*\mathcal{F} \otimes Rg^!\mathcal{G} \to Rg^!(\mathcal{F} \otimes \mathcal{G})$, obtained via adjunction from the map

$$Rg_!(g^*\mathcal{F} \otimes Rg^!\mathcal{G}) \cong \mathcal{F} \otimes Rg_!Rg^!\mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$$

where the first isomorphism is the projection formula.

Suppose now that $f : X \to Y$ is a map of k-varieties, with Y of pure dimension d. Suppose that Y is *rationally smooth* in the sense that ω_Y is lisse; if this happens, then in fact $\omega_Y \cong \Lambda[2d]$. Let $g = (\mathrm{id}, f) : X \to X \times Y$ be the graph of f, with $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the natural projection. Then

$$\Lambda = Rg^!R\pi_X^!\Lambda \cong Rg^!\pi_Y^*\omega_Y \cong Rg^!\Lambda[2d],$$

so in particular $Rg^!\Lambda \cong \Lambda[-2d]$. With this in mind, we specialize the discussion of the previous paragraph to the situation where $T = X, S = X \times Y$, $\mathcal{F} =$

$\mathcal{F}_X \boxtimes \mathcal{F}_Y$ for some arbitrary $\mathcal{F}_X \in D(X)$ and $\mathcal{F}_Y \in D(Y)$, and $\mathcal{G} = \Lambda$. In this situation, $g^*\mathcal{F} = \mathcal{F}_X \otimes f^*\mathcal{F}_Y$ and $Rg^!\Lambda = \Lambda[-2d]$, so γ gives a map

$$\mathcal{F}_X \otimes f^*\mathcal{F}_Y[-2d] \to Rg^!(\mathcal{F}_X \boxtimes \mathcal{F}_Y).$$

When $\mathcal{F}_X$ and $\mathcal{F}_Y$ both lie in D_c^b, we can rewrite the target of this map via the isomorphisms

$$\begin{aligned} Rg^!(\mathcal{F}_X \boxtimes \mathcal{F}_Y) &\cong Rg^!\mathbf{D}\mathbf{D}(\mathcal{F}_X \boxtimes \mathcal{F}_Y) \\ &\cong \mathbf{D}g^*\mathbf{D}(\mathcal{F}_X \boxtimes \mathcal{F}_Y) \\ &\cong \mathbf{D}g^*(\mathbf{D}\mathcal{F}_X \boxtimes \mathbf{D}\mathcal{F}_Y) \\ &= \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y). \end{aligned}$$

In other words, we get a map

$$\beta_f : \mathcal{F}_X \otimes f^*\mathcal{F}_Y[-2d] \to \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y).$$

Definition 1. Let $f : X \to Y$ be as above. A complex $\mathcal{F}_X \in D_c^b(X)$ is *f-locally acyclic (f-LA)* if for all $\mathcal{F}_Y \in D_c^b(Y)$, the map

$$\beta_f : \mathcal{F}_X \otimes f^*\mathcal{F}_Y[-2d] \to \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y)$$

is an isomorphism. If $b'^*\mathcal{F}_X$ is f'-locally acyclic for any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth, then $\mathcal{F}_X$ is *f-universally locally acyclic (f-ULA).*

Note that the subcategories of f-LA or f-ULA complexes in $D_c^b(X)$ are stable under shifts, cones, finite direct sums, and passing to direct summands.

Proposition 2. *A complex $\mathcal{F} \in D_c^b(X)$ is f-LA if and only if the Verdier dual $\mathbf{D}\mathcal{F}$ is f-LA. Likewise for f-ULA.*

Proof. Shifting both sides of the map β_f by $[d]$, we get a map

$$\beta_f^0 : \mathcal{F}_X \otimes f^*\mathcal{F}_Y[-d] \to \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y[-d]).$$

One then checks (with pain) that this map is symmetric, in the sense that the adjoint map

$$\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y[-d] \to \mathbf{D}(\mathcal{F}_X \otimes f^*\mathcal{F}_Y[-d]) = \mathbf{D}(\mathbf{D}\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathbf{D}\mathcal{F}_Y[-d])$$

is also an instance of the map β_f^0, but with $\mathcal{F}_X$ and $\mathcal{F}_Y$ replaced by their Verdier duals. Now use that $\mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\mathbf{D}\mathcal{G} \to \mathbf{D}\mathcal{F}$ is an isomorphism, for any map $\mathcal{F} \to \mathcal{G}$ in $D(X)$. $\square$

Proposition 3. *Let $f : X \to Y$ be as above, and suppose that $p : X' \to X$ is a proper map. If $\mathcal{F}_{X'}$ is $f \circ p$-LA, then $Rp_*\mathcal{F}_{X'}$ is f-LA. Likewise for ULA.*

Proof. By assumption, the map

$$\beta_{f\circ p} : \mathcal{F}_{X'} \otimes p^* f^* \mathcal{F}_Y[-2d] \to \mathbf{D}(\mathbf{D}\mathcal{F}_{X'} \otimes p^* f^* \mathbf{D}\mathcal{F}_Y)$$

is an isomorphism for any $\mathcal{F}_Y$. Applying Rp_* to both sides, we get an isomorphism

$$Rp_*\beta_{f\circ p} : Rp_*(\mathcal{F}_{X'} \otimes p^* f^* \mathcal{F}_Y[-2d]) \to Rp_*\mathbf{D}(\mathbf{D}\mathcal{F}_{X'} \otimes p^* f^* \mathbf{D}\mathcal{F}_Y).$$

Using the projection formula, the left-hand side can be rewritten as $Rp_*\mathcal{F}_{X'} \otimes f^*\mathcal{F}_Y[-2d]$. Similarly, the right-hand side can be rewritten as

$$\begin{aligned} Rp_*\mathbf{D}(\mathbf{D}\mathcal{F}_{X'} \otimes p^* f^* \mathbf{D}\mathcal{F}_Y) &\cong \mathbf{D}Rp_*(\mathbf{D}\mathcal{F}_{X'} \otimes p^* f^* \mathbf{D}\mathcal{F}_Y) \\ &\cong \mathbf{D}(Rp_*\mathbf{D}\mathcal{F}_{X'} \otimes f^* \mathbf{D}\mathcal{F}_Y) \\ &\cong \mathbf{D}(\mathbf{D}Rp_*\mathcal{F}_{X'} \otimes f^* \mathbf{D}\mathcal{F}_Y) \end{aligned}$$

where we used the projection formula along with the commutation $Rp_*\mathbf{D} = \mathbf{D}Rp_*$ (which again relies on properness). Thus $Rp_*\beta_{f\circ p}$ induces an isomorphism

$$Rp_*\mathcal{F}_{X'} \otimes f^*\mathcal{F}_Y[-2d] \to \mathbf{D}(\mathbf{D}Rp_*\mathcal{F}_{X'} \otimes f^*\mathbf{D}\mathcal{F}_Y),$$

and one can check (with pain) that this map actually coincides with the map β_f. □

Corollary 4. *Let $f : X \to Y$ be as above, and suppose there is a proper surjective map $p : X' \to X$ such that $X' \to Y$ is smooth. Then the intersection complex IC_X is f-ULA.*

Proof. Up to shifts, IC_X is a direct summand of $Rp_*\Lambda$, so this follows from the previous proposition. □

Lemma 5. *Fix an integer $c \geq 0$. Let X be any variety, and let $i : V \to X$ be an immersion which (locally on X) factors as $V \to U \to X$ where $U \to X$ is an open immersion and $V \to U$ is a closed immersion whose ideal sheaf can be generated (locally on the affine pieces of some covering of U) by $\leq c$ elements. Then $Ri^![c] : D_c^b(X) \to D_c^b(V)$ is right t-exact for the perverse t-structure, and $i^*[-c] : D_c^b(X) \to D_c^b(V)$ is left t-exact.*

The somewhat awkward condition on i holds, for example, if $i : V \to X$ is a regular immersion of pure codimension c, but it also holds more generally: in particular, unlike the condition of being a regular immersion, it is preserved under arbitrary base change.

Proof. Since $i^*[-c] \cong \mathbf{D}Ri^![c]\mathbf{D}$, and Verdier duality exchanges ${}^{\mathfrak{p}}D_c^{\leq 0}$ and ${}^{\mathfrak{p}}D_c^{\geq 0}$, it suffices to prove the claim for $Ri^![c]$. We easily reduce to the case where $X = \operatorname{Spec} A$ is affine and $i : V \to X$ is a closed immersion associated with an ideal $I \subset A$ generated by $\leq c$ elements. Let $j : U \to X$ be the complementary open immersion, so U can be covered by $\leq c$ open affine subvarieties. Looking at the distinguished triangle $i_*Ri^! \to \mathrm{id} \to Rj_*j^*$, it suffices to prove that Rj_* carries ${}^{\mathfrak{p}}D_c^{\leq 0}(U)$ into ${}^{\mathfrak{p}}D_c^{\leq c-1}(X)$. This is a standard consequence of Artin's cohomological dimension bound for affine morphisms, cf. [BBDG, 4.2.3]. □

Proposition 6. *Let $f : X \to Y$ be as before, and let $\mathcal{F}_X \in D_c^b(X)$ be an f-locally acyclic complex. Let $i : Z \to Y$ be a regular immersion of pure codimension c with Z rationally smooth, with pullback $\tilde{i} : V = X \times_Y Z \to X$. Let $\tilde{f} : V \to Z$ be the pullback of f along $Z \to Y$. Then there is a natural isomorphism $\tilde{i}^*\mathcal{F}_X[-c] \xrightarrow{\sim} R\tilde{i}^!\mathcal{F}_X[c]$, and $\tilde{i}^*\mathcal{F}_X$ is $\tilde{f}$-locally acyclic.*

Moreover, if $\mathcal{F}_X$ is perverse, then so are $\tilde{i}^*\mathcal{F}_X[-c]$ and $R\tilde{i}^!\mathcal{F}_X[c]$.

Note that in general, $\tilde{i}$ may not be a regular immersion, and we don't know anything about the codimension of V in X.

Proof. In the definition of an f-locally acyclic complex, take $\mathcal{F}_Y = Ri_*\Lambda$. The source of the map β_f is then $\mathcal{F}_X \otimes f^*Ri_*\Lambda[-2\mathrm{dim}Y]$. To understand the target, first note that $\mathbf{D}\mathcal{F}_Y = i_!\omega_Z \simeq i_!\Lambda[2\mathrm{dim}Z]$, using that Z is rationally smooth, so then $f^*\mathbf{D}\mathcal{F}_Y = \tilde{i}_!\Lambda[2\mathrm{dim}Z]$. The target of β_f can then be rewritten through the series of isomorphisms

$$\begin{aligned}\mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{F}_Y) &\cong \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes \tilde{i}_!\Lambda[2\mathrm{dim}Z]) \\ &\cong \mathbf{D}(\tilde{i}_!\tilde{i}^*\mathbf{D}\mathcal{F}_X[2\mathrm{dim}Z]) \\ &\cong R\tilde{i}_*R\tilde{i}^!\mathbf{D}\mathbf{D}\mathcal{F}_X[-2\mathrm{dim}Z] \\ &\cong R\tilde{i}_*R\tilde{i}^!\mathcal{F}_X[-2\mathrm{dim}Z].\end{aligned}$$

Here we used the projection formula in the second line, the identification $\mathbf{D}h_!h^* \cong Rh_*Rh^!\mathbf{D}$ (which holds for any immersion h) in the third line, and biduality in the fourth line. Since $\mathcal{F}_X$ is f-locally acyclic by assumption, β_f can now be interpreted as an isomorphism

$$\mathcal{F}_X \otimes f^*Ri_*\Lambda[-2\mathrm{dim}Y] \xrightarrow{\sim} R\tilde{i}_*R\tilde{i}^!\mathcal{F}_X[-2\mathrm{dim}Z].$$

Applying $\tilde{i}^*$ to both sides of this isomorphism, the left side collapses to $\tilde{i}^*\mathcal{F}_X[-2\mathrm{dim}Y]$ and the right side collapses to $R\tilde{i}^!\mathcal{F}_X[-2\mathrm{dim}Z]$. Shifting by $[\mathrm{dim}Y + \mathrm{dim}Z]$ then gives the desired isomorphism $\tilde{i}^*\mathcal{F}_X[-c] \xrightarrow{\sim} R\tilde{i}^!\mathcal{F}_X[c]$.

Next, take $\mathcal{F}_Y = Ri_*\mathcal{F}_Z$ for some arbitrary $\mathcal{F}_Z \in D_c^b(Z)$. By a slight generalization of the argument used in the previous paragraph, we can rewrite the target of β_f as $R\tilde{i}_*\mathbf{D}(\tilde{i}^*\mathbf{D}\mathcal{F}_X \otimes \tilde{f}^*\mathbf{D}\mathcal{F}_Z)$. But now we can write

$$\begin{aligned}\tilde{i}^*\mathbf{D}\mathcal{F}_X &\cong \mathbf{D}R\tilde{i}^!\mathcal{F}_X \\ &\cong \mathbf{D}(\tilde{i}^*\mathcal{F}_X[-2c])\end{aligned}$$

by the first part of the proposition, so we can interpret β_f as an isomorphism

$$\mathcal{F}_X \otimes f^*Ri_*\mathcal{F}_Z[-2d] \to R\tilde{i}_*\mathbf{D}(\mathbf{D}\tilde{i}^*\mathcal{F}_X \otimes \tilde{f}^*\mathbf{D}\mathcal{F}_Z)[-2c].$$

Applying $\tilde{i}^*$ to both sides and shifting by $[2c]$, this becomes the desired isomorphism

$$\tilde{i}^*\mathcal{F}_X \otimes \tilde{f}^*\mathcal{F}_Z[2c-2d] \to \mathbf{D}(\mathbf{D}\tilde{i}^*\mathcal{F}_Z \otimes \tilde{f}^*\mathbf{D}\mathcal{F}_Z).$$

(Again, you can check, if you dare, that this map actually coincides with the map $\beta_{\tilde{f}}$.) Thus $\tilde{i}^*\mathcal{F}_X$ is $\tilde{f}$-LA.

Finally, note that $\tilde{i} : V \to X$ satisfies the hypotheses of the previous lemma, so if $\mathcal{F}_X$ is perverse, then $\tilde{i}^*\mathcal{F}_X[-c]$ lies in ${}^{\mathfrak{p}}D_c^{\geq 0}(V)$ and $R\tilde{i}^!\mathcal{F}_X[c]$ lies in ${}^{\mathfrak{p}}D_c^{\leq 0}(V)$. However, by the previous part of the proposition, these two complexes are isomorphic, so we conclude that they both lie in the heart of the perverse t-structure, as desired. □

Proposition 7. *Let $f : X \to Y$ be as above, and let $\mathcal{F}_X \in D_c^b(X)$ be an f-locally acyclic complex. Suppose also that $\mathcal{F}_X$ is perverse. Then the functor*

$$D_c^b(Y) \to D_c^b(X)$$
$$\mathcal{G} \mapsto \mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim} Y]$$

is t-exact for the perverse t-structures.

Proof. It suffices to show that for any perverse sheaf $\mathcal{G} \in D_c^b(Y)$, the complex $\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim} Y]$ satisfies the support condition in the definition of a perverse sheaf. Indeed, local acyclicity guarantees that

$$\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim} Y] \cong \mathbf{D}(\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{G}[-\mathrm{dim} Y]),$$

and $\mathbf{D}\mathcal{F}_X$ is again f-LA and perverse, so $\mathbf{D}\mathcal{F}_X \otimes f^*\mathbf{D}\mathcal{G}[-\mathrm{dim} Y]$ also satisfies the support condition. The Verdier dual then transforms this into the cosupport condition for $\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim} Y]$.

Fix a perverse sheaf $\mathcal{G} \in D_c^b(Y)$, and fix a stratification $Y = \coprod Y_m$ by smooth locally closed subvarieties $Y_m \subset Y$ of pure codimension m such that all the (ordinary) cohomology sheaves of each restriction $\mathcal{G}|Y_m$ are lisse. Set $X_m = X \times_Y Y_m$. Fix an integer n. We are going to estimate the dimension of

$$S_m \overset{\mathrm{def}}{=} X_m \cap \mathrm{supp}\mathcal{H}^n(\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim} Y])$$

for each m. This set is clearly equal to

$$\bigcup_j f^{-1}(\mathrm{supp}\mathcal{H}^{j-\mathrm{dim} Y}(\mathcal{G}|Y_m)) \cap (\mathrm{supp}\mathcal{H}^{n-j}(\mathcal{F}_X|X_m)).$$

Since $\mathcal{G}$ is perverse and constructible for the stratification by the Y_m's, $\mathcal{H}^k(\mathcal{G}|Y_m)$ is nonvanishing only for $k \in [-\mathrm{dim} Y, -\mathrm{dim} Y + m]$, and $\mathrm{supp}\mathcal{H}^k(\mathcal{G}|Y_m)$ is an open-closed subset of Y_m for any k. In particular, only j's in the interval $[0, m]$ contribute in the previous description of S_m, and we get an inclusion

$$S_m \subseteq \bigcup_{j \in [0,m]} \mathrm{supp}\mathcal{H}^{n-j}(\mathcal{F}_X|X_m).$$

Now, by the previous proposition, each restriction $\mathcal{F}_X|X_m$ is of the form $\mathcal{P}_m[m]$ for some perverse sheaf $\mathcal{P}_m$ on X_m. Then for any $j \in [0, m]$ we have

$$\begin{aligned}\dim \mathrm{supp}\mathcal{H}^{n-j}(\mathcal{F}_X|X_m) &= \dim \mathrm{supp}\mathcal{H}^{n-j}(\mathcal{P}_m[m]) \\ &= \dim \mathrm{supp}\mathcal{H}^{n+m-j}(\mathcal{P}_m) \\ &\leq -n + j - m \\ &\leq -n,\end{aligned}$$

where we used the support condition for $\mathcal{P}_m$ in the third line and the containment $j \in [0, m]$ in the fourth line. Therefore $\dim S_m \leq -n$ for any m. Since

$$\mathrm{supp}\mathcal{H}^n(\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim}Y]) = \coprod_m S_m$$

by definition, we conclude that $\dim \mathrm{supp}\mathcal{H}^n(\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim}Y]) \leq -n$ for any n, which is exactly the support condition for $\mathcal{F}_X \otimes f^*\mathcal{G}[-\mathrm{dim}Y]$. □

The next proposition recovers (most of) a result of Reich, cf. [R, Prop. IV.2.8], and in fact is slightly more general. Reich's proof uses the notions of unipotent nearby cycles and unipotent vanishing cycles; our argument avoids this machinery.

Proposition 8. *Let $f : X \to Y$ be as above, and let $D \subset Y$ be a rationally smooth effective Cartier divisor; set $Z = X \times_Y D$ and $U = f^{-1}(Y \smallsetminus D)$, and let $i : Z \to X$ and $j : U \to X$ be the evident immersions. Let $\mathcal{F} \in D^b_c(X)$ be an f-locally acyclic complex, and suppose that $\mathcal{F}|U$ is perverse. Then $\mathcal{F}$ is perverse, and in fact $\mathcal{F} \simeq j_{!*}\mathcal{F}$.*

In particular, if $\mathcal{G} \in D^b_c(U)$ is perverse, there is (up to isomorphism) at most one f-locally acyclic complex on X extending $\mathcal{G}$, and such an extension is automatically perverse.

Proof. By construction, j is affine, so $j_!$ and Rj_* are t-exact for the perverse t-structures. Therefore, the third term in the distinguished triangle

$$i_*Ri^!\mathcal{F} \to \mathcal{F} \to Rj_*j^*\mathcal{F} \xrightarrow{[1]}$$

is perverse; moreover, the first term is isomorphic to $i_*i^*\mathcal{F}[-2]$ by Proposition 6. Taking the long exact sequence of perverse cohomology sheaves then gives an isomorphism ${}^{\mathfrak{p}}\mathcal{H}^j(i_*i^*\mathcal{F}) \simeq {}^{\mathfrak{p}}\mathcal{H}^{j+2}(\mathcal{F})$ for any $j \geq 0$ or $j \leq -3$. Arguing analogously with the distinguished triangle

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^* \xrightarrow{[1]}$$

gives an isomorphism ${}^{\mathfrak{p}}\mathcal{H}^j(\mathcal{F}) \simeq {}^{\mathfrak{p}}\mathcal{H}^j(i_*i^*\mathcal{F})$ for any $j \geq 1$ or $j \leq -2$. Combining these isomorphisms, we get isomorphisms ${}^{\mathfrak{p}}\mathcal{H}^j(\mathcal{F}) \simeq {}^{\mathfrak{p}}\mathcal{H}^{j+2}(\mathcal{F})$ for every $j \geq 1$, and ${}^{\mathfrak{p}}\mathcal{H}^j(\mathcal{F}) \simeq {}^{\mathfrak{p}}\mathcal{H}^{j-2}(\mathcal{F})$ for every $j \leq -1$. Therefore, we must have ${}^{\mathfrak{p}}\mathcal{H}^j(\mathcal{F}) = 0$ for every $j \neq 0$, since otherwise $\mathcal{F}$ would have infinitely many nonzero perverse cohomology sheaves, contradicting the boundedness of $\mathcal{F}$. This shows that $\mathcal{F}$ is perverse. Applying Proposition 6 again, we deduce that $i^*\mathcal{F}[-1]$ and $Ri^!\mathcal{F}[1]$ are both perverse as well. In particular, $i^*\mathcal{F} \in {}^{\mathfrak{p}}D^{\leq -1}(Z)$ and $Ri^!\mathcal{F} \in {}^{\mathfrak{p}}D^{\geq 1}(Z)$, and these conditions uniquely characterize the intermediate extension of $\mathcal{F}|U$. □

References

[BBDG] A. Beilinson, J. Bernstein, P. Deligne, and O. Gabber, *Faisceaux Pervers*

[BG] A. Braverman and D. Gaitsgory, *Geometric Eisenstein series*

[R] R. Reich, *Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian*

[Ri] T. Richarz, *A new approach to the geometric Satake equivalence*